

Alsafwa University College

Department of Computer Engineering Techniques

MATHEMATICS

One Stage

Chapter One

Matrices

Definition:

A matrices is a set of real or complex numbers (or elements) arranged in rows and columns to form rectangular array. A matrix having **m** rows and **n** columns is called (**m**×**n**) matrix and is referred to as having order (**m**×**n**) .

A matrix is in dictating by writing the array within brackets.

$$\begin{bmatrix} a_{11} & a_{12} & \dots & a_{1n} \\ a_{21} & a_{22} & \dots & a_{2n} \\ a_{m1} & a_{m2} & \dots & a_{mn} \end{bmatrix}$$

A horizontal line of elements is called row, and a vertical line is called a column.

For example:

$$\begin{bmatrix} 5 & 7 & 2 \\ 6 & 3 & 8 \end{bmatrix} \text{ is } (2 \times 3) \text{ matrix}$$

2 the numbers of rows and 3 the number of columns.

Types of Matrices:

- **Row matrix:** consists of 1 row only, for example $[4 \ 3 \ 7 \ 2]$ is a row matrix of - order (1×4) .

- **Column matrix:** consists of 1 column only, for example

$$\begin{bmatrix} 6 \\ 3 \\ 8 \end{bmatrix} \text{ is a column matrix of order } (3 \times 1).$$

- **Square matrix:** is a matrix in which the number of rows (**m**) equals the number of columns (**n**) for example

$$S = \begin{bmatrix} 2 & 0 & 5 \\ 7 & 8 & 7 \\ 6 & 7 & 5 \end{bmatrix}$$

- **Rectangular matrix:** A matrix of any size $(m \times n)$ and this includes square matrices as a special case.

- **Diagonal matrix:** is a square matrix with all elements zero except those on the main diagonal. For example

$$D = \begin{bmatrix} -1 & 0 & 0 \\ 0 & 4 & 0 \\ 0 & 0 & 7 \end{bmatrix}$$

- **Unit matrix:** is a diagonal matrix in which the elements on the main diagonal are all unity. For example

$$I = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

-Null (zero) matrix: is one whose elements are zero. For example

$$O = \begin{bmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

-Vector matrix: is a matrix with only one row or column. Its entries are called the component of the vector.

Some Operations on Matrices:

1- Equality of matrices:

Tow matrices A and B are equal if and only if they have the same size and corresponding entries are equal, matrices that are not equal are called different.

Example: let $A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 3 & -1 \end{bmatrix}$

$A=B$ if and only if: $a_{11} = 4, a_{12} = 0, a_{21} = 3, a_{22} = -1$

Example: If $A = \begin{bmatrix} 2 & 3 & a \\ b & 6 & 1 \end{bmatrix}$, $B = \begin{bmatrix} 2 & 3 & 9 \\ -3 & 6 & 1 \end{bmatrix}$ and $C = \begin{bmatrix} 2 & 3 & 9 \\ -3 & 6 & 1 \\ 0 & 0 & 0 \end{bmatrix}$

Then $A = B$ if $a = 9$ and $b = -3$ but $A \neq C$ and $B \neq C$.

2-Addition and Subtraction of Matrices:

To be added or subtracted, two matrices must be of the same order. The sum or difference is then determined by adding or subtracting corresponding elements.

For example:

$$\begin{bmatrix} 4 & 2 & 3 \\ 5 & 7 & 6 \end{bmatrix} + \begin{bmatrix} 1 & 8 & 9 \\ 3 & 5 & 4 \end{bmatrix} = \begin{bmatrix} 4+1 & 2+8 & 3+9 \\ 5+3 & 7+5 & 6+4 \end{bmatrix} = \begin{bmatrix} 5 & 10 & 12 \\ 8 & 12 & 10 \end{bmatrix}$$

$$\begin{bmatrix} 6 & 5 & 12 \\ 9 & 4 & 8 \end{bmatrix} - \begin{bmatrix} 3 & 7 & 1 \\ 2 & 10 & -5 \end{bmatrix} = \begin{bmatrix} 6-3 & 5-7 & 12-1 \\ 9-2 & 4-10 & 8+5 \end{bmatrix} = \begin{bmatrix} 3 & -2 & 11 \\ 7 & -6 & 13 \end{bmatrix}$$

3-Multiplication of Matrices:

Tow matrices can be multiplied together only when the number of columns in the first is equal to the number of rows in the second, for example:

$$\text{If } A = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \text{ and } B = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

Then:

$$A \cdot B = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \end{bmatrix} \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix} = \begin{bmatrix} a_{11}b_1 + a_{12}b_2 + a_{13}b_3 \\ a_{21}b_1 + a_{22}b_2 + a_{23}b_3 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{bmatrix}$ and $B = \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix}$, find $A \cdot B$

Sol:

$$A \cdot B = \begin{bmatrix} 4 & 7 & 6 \\ 2 & 3 & 1 \end{bmatrix} \cdot \begin{bmatrix} 8 \\ 5 \\ 9 \end{bmatrix} = \begin{bmatrix} 4*8 + 7*5 + 6*9 \\ 2*8 + 3*5 + 1*9 \end{bmatrix} = \begin{bmatrix} 32 + 35 + 54 \\ 16 + 15 + 9 \end{bmatrix} = \begin{bmatrix} 121 \\ 40 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{bmatrix}$ and $B = \begin{bmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{bmatrix}$, find $A \cdot B$

Sol:

$$A \cdot B = \begin{bmatrix} 1 & 5 \\ 2 & 7 \\ 3 & 4 \end{bmatrix} \cdot \begin{bmatrix} 8 & 4 & 3 & 1 \\ 2 & 5 & 8 & 6 \end{bmatrix} = \begin{bmatrix} 8+10 & 4+25 & 3+40 & 1+30 \\ 16+14 & 8+35 & 6+56 & 2+42 \\ 24+8 & 12+20 & 9+32 & 3+24 \end{bmatrix}$$

$$= \begin{bmatrix} 18 & 29 & 43 & 31 \\ 30 & 43 & 62 & 44 \\ 32 & 32 & 41 & 27 \end{bmatrix}$$

Note: If A is an ($m \times n$) matrix and B is ($n \times m$) matrix, then products $\mathbf{A} \cdot \mathbf{B}$ and $\mathbf{B} \cdot \mathbf{A}$ are possible.

Example: If $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix}$, $B = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix}$, find $A \cdot B$ and $B \cdot A$

Sol:

$$A \cdot B = \begin{bmatrix} 7+16+27 & 10+22+36 \\ 28+40+54 & 40+55+72 \end{bmatrix} = \begin{bmatrix} 50 & 68 \\ 122 & 167 \end{bmatrix}$$

$$B \cdot A = \begin{bmatrix} 7 & 10 \\ 8 & 11 \\ 9 & 12 \end{bmatrix} \begin{bmatrix} 1 & 2 & 3 \\ 4 & 5 & 6 \end{bmatrix} = \begin{bmatrix} 7+40 & 14+50 & 21+60 \\ 8+44 & 16+55 & 24+66 \\ 9+48 & 18+60 & 27+72 \end{bmatrix}$$

$$= \begin{bmatrix} 47 & 64 & 81 \\ 52 & 71 & 90 \\ 57 & 78 & 99 \end{bmatrix}$$

Note: $\mathbf{A} \cdot \mathbf{B} \neq \mathbf{B} \cdot \mathbf{A}$ Multiplication is not commutative.

Properties of Matrix Operations:

If A and B, C are $(m \times n)$ matrices, O is zero matrix and K, R are any scalars then:

- $A + B = B + A$
- $(A + B) + C = A + (B + C)$
- $A + O = A, A + (-A) = O$
- $K(RA) = (KR)A, (K + R)A = KA + RA$
- $I A = A, O A = O, R O = O$

Transpose of Matrix:

If the rows and columns of matrix are interchanged, then the new matrix is called the transpose of the original matrix. If A^T is the transpose of matrix A , then

$A \neq A^T$, For example:

$$A = \begin{bmatrix} 4 & 6 \\ 7 & 9 \\ 2 & 5 \end{bmatrix} \rightarrow A^T = \begin{bmatrix} 4 & 7 & 2 \\ 6 & 9 & 5 \end{bmatrix}$$

Example: If $A = \begin{bmatrix} 2 & 7 & 6 \\ 3 & 1 & 5 \end{bmatrix}$ and $B = \begin{bmatrix} 4 & 0 \\ 3 & 7 \\ 1 & 5 \end{bmatrix}$, find $A \cdot B$ and $(A \cdot B)^T$.

Sol:

$$A \cdot B = \begin{bmatrix} 35 & 79 \\ 20 & 32 \end{bmatrix}, \quad (A \cdot B)^T = \begin{bmatrix} 35 & 20 \\ 79 & 32 \end{bmatrix}$$

Special Matrices:

Square matrix is a matrix of order ($m \times m$). A square matrix is **symmetric** if

$a_{ij} = a_{ji}$ means $A = A^T$ for example

$$A = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{bmatrix}, \quad A^T = \begin{bmatrix} 1 & 2 & 5 \\ 2 & 8 & 9 \\ 5 & 9 & 4 \end{bmatrix}$$

A square matrix is **skew-symmetric** if $a_{ij} = -a_{ji}$ means $A \neq A^T$ for example

$$A = \begin{bmatrix} 0 & 2 & 5 \\ -2 & 0 & 9 \\ -5 & -9 & 0 \end{bmatrix}, \quad A^T = \begin{bmatrix} 0 & -2 & -5 \\ 2 & 0 & -9 \\ 5 & 9 & 0 \end{bmatrix}$$

Example: Given that $A = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix}$ determine A^T and $A \cdot A^T$.

Sol:

$$A = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix}, \quad A^T = \begin{bmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{bmatrix}$$

$$A \cdot A^T = \begin{bmatrix} 4 & 2 & 6 \\ 1 & 8 & 7 \end{bmatrix} \begin{bmatrix} 4 & 1 \\ 2 & 8 \\ 6 & 7 \end{bmatrix} = \begin{bmatrix} 16 + 4 + 36 & 4 + 16 + 42 \\ 4 + 16 + 42 & 4 + 64 + 49 \end{bmatrix} = \begin{bmatrix} 56 & 62 \\ 62 & 117 \end{bmatrix}$$

Determinant of a Square Matrix:

The determinant of a square matrix is the determinant having the same elements as those of the matrix. For example

$$A = \begin{bmatrix} 5 & 2 & 1 \\ 0 & 6 & 3 \\ 8 & 4 & 7 \end{bmatrix} \quad \text{then the det of A is given by:}$$

$$\begin{aligned} |A| &= 5(42 - 12) - 2(0 - 24) + 1(0 - 48) = 5(30) - 2(-24) + 1(-48) \\ &= 150 + 48 - 48 = 150 \end{aligned}$$

Note that the transpose of $A^T = \begin{bmatrix} 5 & 0 & 8 \\ 2 & 6 & 4 \\ 1 & 3 & 7 \end{bmatrix}$

And the determinate of A^T is

$$|A^T| = 5(42 - 12) - 0(14 - 4) + 8(6 - 6) = 5(30) = 150$$

Note that: $\det(A) = \det(A^T)$.

Cofactors:

If A is square matrix, the determinates of its element will be:

$$A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}$$

$$\det(A) = 2(0 - 24) - 3(0 - 6) + 5(16 - 1) = 45$$

The minor of element 2 is $+ \begin{bmatrix} 1 & 6 \\ 4 & 0 \end{bmatrix} = 0 - 24 = -24$

Similarly the cofactor of element 3 is $- \begin{bmatrix} 4 & 6 \\ 1 & 0 \end{bmatrix} = -(0 - 6) = 6$

The cofactor of element 5 is $+ \begin{bmatrix} 4 & 1 \\ 1 & 4 \end{bmatrix} = +(16 - 1) = 1$

The cofactor of element 4 is $- \begin{bmatrix} 3 & 5 \\ 4 & 0 \end{bmatrix} = -(0 - 20) = 20$

The cofactor of element 1 is $+ \begin{bmatrix} 2 & 5 \\ 1 & 0 \end{bmatrix} = +(0 - 5) = -5$

The cofactor of element 6 is $- \begin{bmatrix} 2 & 3 \\ 1 & 4 \end{bmatrix} = -(8 - 3) = -5$

The cofactor of element 1 is $+ \begin{bmatrix} 3 & 5 \\ 1 & 6 \end{bmatrix} = +(18 - 5) = 13$

The cofactor of element 4 is $- \begin{bmatrix} 2 & 5 \\ 4 & 6 \end{bmatrix} = -(12 - 20) = 8$

The cofactor of element 0 is $+ \begin{bmatrix} 2 & 3 \\ 4 & 1 \end{bmatrix} = +(2 - 12) = -10$

The cofactor matrix C is

$$C = \begin{bmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{bmatrix}$$

And the transpose of C is

$$C^T = \begin{bmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{bmatrix}$$

Where C^T is called the adjoint of matrix $A = adj A$.

Inverse of a Square Matrix:

If $A = \begin{bmatrix} 2 & 3 & 5 \\ 4 & 1 & 6 \\ 1 & 4 & 0 \end{bmatrix}$ then the inverse of A is A^{-1} and given by:

$$A^{-1} = \frac{\text{adj } A}{\det A}$$

$$\det A = |A| = 2(0 - 24) - 3(0 - 6) + 5(16 - 1) = 45$$

The cofactor matrix C is:

$$C = \begin{bmatrix} -24 & 6 & 15 \\ 20 & -5 & -5 \\ 13 & 8 & -10 \end{bmatrix} \text{ and } C^T = \text{adj } A \begin{bmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{1}{45} \begin{bmatrix} -24 & 20 & 13 \\ 6 & -5 & 8 \\ 15 & -5 & -10 \end{bmatrix} = \begin{bmatrix} -24/45 & 20/45 & 13/45 \\ 6/45 & -5/45 & 8/45 \\ 15/45 & -5/45 & -10/45 \end{bmatrix}$$

Example: Find the inverse of the given matrix $A = \begin{bmatrix} 1 & 2 & 3 \\ 4 & 1 & 5 \\ 6 & 0 & 2 \end{bmatrix}$

Sol:

$$\det A = |A| = 1(2 - 0) - 2(8 - 30) + 3(0 - 6) = 28$$

The cofactor of element 1 is $+(2 - 0) = 2$

The cofactor of element 4 is $-(4 - 0) = -4$

The cofactor of element 6 is $+(10 - 3) = 7$

The cofactor of element 2 is $-(8 - 30) = 22$

The cofactor of element 1 is $+(2 - 18) = -16$

The cofactor of element 0 is $-(5 - 12) = 7$

The cofactor of element 3 is $+(0 - 6) = -6$

The cofactor of element 5 is $-(0 - 12) = 12$

The cofactor of element 2 is $+(1 - 8) = -7$

The cofactor matrix C is

$$C = \begin{bmatrix} 2 & 22 & -6 \\ -4 & -16 & 12 \\ 7 & 7 & -7 \end{bmatrix} \quad \text{and} \quad \text{adj } A = C^T = \begin{bmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{bmatrix}$$

$$A^{-1} = \frac{\text{adj } A}{\det A} = \frac{1}{28} \begin{bmatrix} 2 & -4 & 7 \\ 22 & -16 & 7 \\ -6 & 12 & -7 \end{bmatrix} = \begin{bmatrix} 2/28 & -4/28 & 7/28 \\ 22/28 & -16/28 & 7/28 \\ -6/28 & 12/28 & -7/28 \end{bmatrix}$$

$$= \begin{bmatrix} 1/14 & -1/7 & 1/4 \\ 11/14 & -8/14 & 1/4 \\ -3/14 & 6/14 & -1/4 \end{bmatrix}$$

Grammar's rule for solving a set of linear equations:

Consider a set of linear equations in three unknowns x, y, z

$$a_{11}x + a_{12}y + a_{13}z = b_1 \quad (1)$$

$$a_{21}x + a_{22}y + a_{23}z = b_2 \quad (2)$$

$$a_{31}x + a_{32}y + a_{33}z = b_3 \quad (3)$$

In matrices notation the system of linear equations may be written as:

$$\begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} b_1 \\ b_2 \\ b_3 \end{bmatrix}$$

The above theorem called Grammar's rule to solve it we put:

$$D = \begin{bmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{bmatrix}, \quad D_1 = \begin{bmatrix} b_1 & a_{12} & a_{13} \\ b_2 & a_{22} & a_{23} \\ b_3 & a_{32} & a_{33} \end{bmatrix}$$

$$D_2 = \begin{bmatrix} a_{11} & b_1 & a_{13} \\ a_{21} & b_2 & a_{23} \\ a_{31} & b_3 & a_{33} \end{bmatrix}, \quad D_3 = \begin{bmatrix} a_{11} & a_{12} & b_1 \\ a_{21} & a_{22} & b_2 \\ a_{31} & a_{32} & b_3 \end{bmatrix}$$

If $D \neq 0$ then the system has unique solution.

$$\therefore x = \frac{D_1}{D}, \quad y = \frac{D_2}{D}, \quad z = \frac{D_3}{D}$$

Example: Use Grammar's rule to solve the system

$$5x - 2y = -1$$

$$2x + 3y = 3$$

Sol:

$$\begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} -1 \\ 3 \end{bmatrix}$$

$$D = \begin{bmatrix} 5 & -2 \\ 2 & 3 \end{bmatrix} = 15 + 4 = 19$$

$$D_1 = \begin{bmatrix} -1 & -2 \\ 3 & 3 \end{bmatrix} = -3 + 6 = 3$$

$$D_2 = \begin{bmatrix} 5 & -1 \\ 2 & 3 \end{bmatrix} = 15 + 2 = 17$$

$$\therefore x = \frac{D_1}{D} = \frac{3}{19}, \quad y = \frac{D_2}{D} = \frac{17}{19}$$

Example: Use Grammar's rule to solve the system

$$x + 2z = 6$$

$$-3x + 4y + 6z = 30$$

$$-x - 2y + 3z = 8$$

Sol:

$$\begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} \begin{bmatrix} x \\ y \\ z \end{bmatrix} = \begin{bmatrix} 6 \\ 30 \\ 8 \end{bmatrix}$$

$$D = \begin{bmatrix} 1 & 0 & 2 \\ -3 & 4 & 6 \\ -1 & -2 & 3 \end{bmatrix} = 1(12 + 12) - 0 + 2(6 + 4) = 24 + 20 = 44$$

$$D_1 = \begin{bmatrix} 6 & 0 & 2 \\ 30 & 4 & 6 \\ 8 & -2 & 3 \end{bmatrix} = 6(12 + 12) + 2(-60 - 32) = 144 - 184 = -40$$

$$D_2 = \begin{bmatrix} 1 & 6 & 2 \\ -3 & 30 & 6 \\ -1 & 8 & 3 \end{bmatrix} = 1(90 - 48) - 6(-9 + 6) + 2(-24 + 30) \\ = 42 + 18 + 12 = 72$$

$$D_3 = \begin{bmatrix} 1 & 0 & 6 \\ -3 & 4 & 30 \\ -1 & -2 & 8 \end{bmatrix} = 1(32 + 60) - 6(6 + 4) = 92 + 60 = 152$$

$$\therefore x = \frac{D_1}{D} = -\frac{40}{44} , \quad y = \frac{D_2}{D} = \frac{72}{44} , \quad z = \frac{D_3}{D} = \frac{152}{44}$$

HOMEWORK

1- Solve the following equations using Grammar's rule:

a) $3x + 8y = 4$

$$3x - y = -13$$

b) $2x + y - z = 2$

$$x - y + z = 7$$

$$2x + 2y + z = 4$$

2- Find $A \cdot A^{-1}$ if the matrix A is given by: $A = \begin{bmatrix} 3 & 0 & 1 \\ 0 & 0 & 2 \\ -1 & 1 & 5 \end{bmatrix}$

3- If $A = \begin{bmatrix} x^2 & 2 & 9 \\ 1+y & 4 & 0 \\ 2 & 3 & 3 \end{bmatrix}$ and $B = \begin{bmatrix} 0 & 2 & 9 \\ 5 & 4 & 0 \\ 2 & 3 & 3 \end{bmatrix}$, find the values of x, y .

Chapter Two

Inequalities and Functions

Definition:

If a and b are real number, then one of the following is true

$$a > b, \quad a < b, \quad a = b$$

If $a > b$ then $-a < b$; $-2 < -1$

If $a > b$ then $\frac{1}{a} < \frac{1}{b}$; $\frac{1}{5} < \frac{1}{4}$

Intervals:

Definition: An interval is a set of numbers x having one of the following

- 1- Open interval: $a < x < b \equiv (a, b)$
- 2- Closed interval: $a \leq x \leq b \equiv [a, b]$
- 3- Half open from the left or half close from the right $a < x \leq b \equiv (a, b]$
- 4- Half close from the left or half open from the right $a \leq x < b \equiv [a, b)$

Notes:

$$1- a < x < \infty \equiv a < x \equiv (a, \infty)$$

$$2- a \leq x < \infty \equiv a \leq x \equiv [a, \infty)$$

$$3- \infty < x < a \equiv x < a \equiv (-\infty, a)$$

$$4- \infty < x \leq a \equiv x \leq a \equiv (-\infty, a]$$

Absolute Value:

Definition: The absolute value of real number x is define as:

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0 \end{cases}$$

Properties of absolute value:

$$1 - |x| = a \text{ if and only if } x = \pm a$$

2- $|x| = |-x|$, a number and its additive inverse or negative have

the same absolute value.

$$3 - |x \cdot y| = |x| \cdot |y| \text{ and } \left| \frac{x}{y} \right| = \frac{|x|}{|y|}$$

$$4 - |-x| = |x|, |a| = \sqrt{a^2} \text{ where } a = \text{scaler.}$$

$$5 - |x \pm y| \leq |x| \pm |y|$$

$$6 - |x| < a \text{ this means } -a < x < a$$

$$7 - |x| \geq a \text{ this means } -a \leq x \leq a$$

$$8 - |x| > a \text{ this means } x < -a \text{ or } x > a$$

$9 - |x| \geq a$ this means $x \leq -a$ or $x \geq a$

Example: Find the absolute value of the following:

$$1) \left| \frac{2x+1}{4} \right| \leq 6 \quad 2) |5x - 2| \geq 1 \quad 3) |2x - 3| \leq 1$$

Sol:

$$1) \left| \frac{2x+1}{4} \right| \leq 6 \rightarrow \left[-6 \leq \frac{2x+1}{4} \leq 6 \right] * 4 \rightarrow -24 \leq 2x + 1 \leq 24$$

$$-1 - 24 \leq 2x + 1 - 1 \leq 24 - 1 \rightarrow [-25 \leq 2x \leq 23] \div 2$$

$$-\frac{25}{2} \leq x \leq \frac{23}{2}$$

$$2) |5x - 2| \geq 1$$

$$5x - 2 \geq 1 \quad \text{or} \quad 5x - 2 \leq -1$$

$$5x - 2 + 2 \geq 1 + 2 \quad \text{or} \quad 5x - 2 + 2 \leq -1 + 2$$

$$5x \geq 3 \quad \text{or} \quad 5x \leq 1$$

$$\frac{5x}{5} \geq \frac{3}{5} \quad \text{or} \quad \frac{5x}{5} \leq \frac{1}{5}$$

$$x \geq \frac{3}{5} \quad \text{or} \quad x \leq \frac{1}{5}$$

$$3) |2x - 3| \leq 1$$

$$|2x - 3| \leq 1 \rightarrow -1 \leq 2x - 3 \leq 1$$

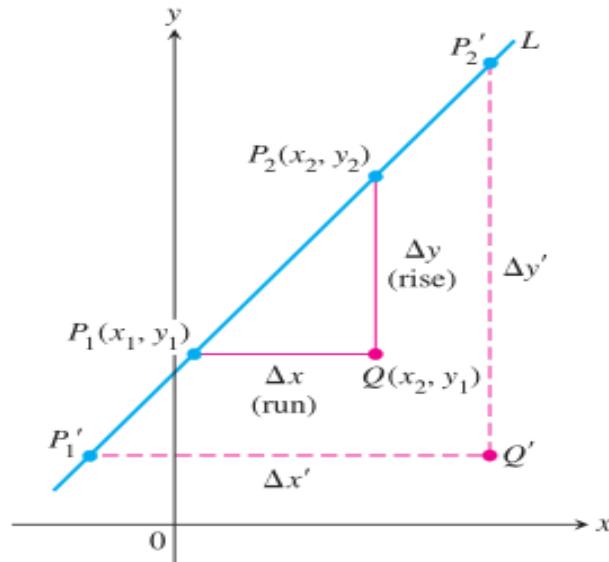
$$-1 + 3 \leq 2x - 3 \leq 1 + 3 \rightarrow [2 \leq 2x \leq 4] \div 2 \rightarrow 1 \leq x \leq 2$$

Functions and their graphs:

Increments and Straight Lines:

When a particle moves from one point in the plane to another, the net changes in its coordinates are called increments. They are calculated by subtracting the coordinates of the starting point from the coordinates of the ending point. If x changes from x_1 to x_2 and y increment changes from y_1 to y_2 in y then the increment in x and y respectively is:

$$\Delta x = x_2 - x_1 \quad \text{and} \quad \Delta y = y_2 - y_1$$



Slope:

Definition: the slope of the nonvertical line $p_1(x_1, y_1)$ and $p_2(x_2, y_2)$

$$m = \frac{\Delta y}{\Delta x} = \frac{y_2 - y_1}{x_2 - x_1} = \tan \theta$$

We can write an equation for a nonvertical straight line L if we know its slope m and the coordinates of one point $p_1(x_1, y_1)$ on it. If $p_1(x_1, y_1)$ is any other point on L, then we can use the two points p_1 and p to compute the slope,

$$m = \frac{y - y_1}{x - x_1} \quad \text{or} \quad y = y_1 + m(x - x_1)$$

This equation is called the point-slope equation of the line that passes through the point $p_1(x_1, y_1)$ and has slope m .

Example: Write an equation for the line through the point (2, 3) with slope $-3/2$.

Sol:

$$y = y_1 + m(x - x_1)$$

We substitute $x_1 = 2$ and $y_1 = 3$ into the point-slope equation and obtain

$$y = 3 - \frac{3}{2}(x - 2) = 3 - \frac{3}{2}x + 3 = -\frac{3}{2}x + 6$$

Example: Write an equation for the line through (-2, -1) and (3, 4).

Sol:

$$m = \frac{y_2 - y_1}{x_2 - x_1} = \frac{4 - (-1)}{3 - (-2)} = \frac{4 + 1}{3 + 2} = \frac{5}{5} = 1$$

We can use this slope with either of the two given points in the point-slope equation:

$$y = y_1 + m(x - x_1) = -1 + 1(x + 2) = x + 1$$

Tangent Line:

The tangent line to the curve at P is the line through P with this slope.

Finding the tangent $y = f(x)$ to the curve at (x, y) by derive the function y with respect to x and then apply :

$$y = y_0 + m(x - x_0)$$

Example: Find the slope of the curve and the tangent line of

$$y = 1 + x^2 \text{ at } (2,5).$$

Sol:

The slope

$$m = \frac{dy}{dx} = 2x \quad \text{at } x = 2 \quad m = 2 * 2 = 4 \quad .$$

Tangent line

$$y = y_0 + m(x - x_0) = 5 + 4(x - 2) = 5 + 4x - 8 = 4x - 3.$$

Graphs of Functions:

If f is a function with domain D , its graph consists of the points in the cartesian plane whose coordinates are the input-output pairs for f .

Example: Graph the function $y = x^2$ over the interval $[-2,2]$.

Sol:

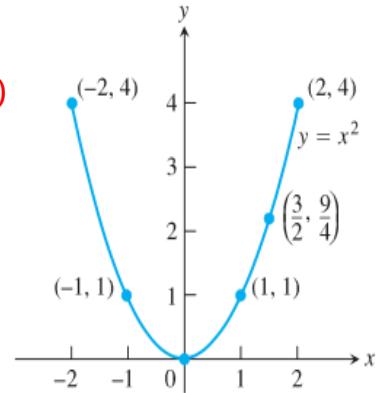
$$we \ put \ x = 0 \rightarrow y = x^2 \rightarrow y = (0)^2 \rightarrow y = 0 \rightarrow (0,0)$$

$$x = 1 \rightarrow y = x^2 \rightarrow y = (1)^2 \rightarrow y = 1 \rightarrow (1,1)$$

$$x = -1 \rightarrow y = x^2 \rightarrow y = (-1)^2 \rightarrow y = 1 \rightarrow (-1,1)$$

$$x = -2 \rightarrow y = x^2 \rightarrow y = (-2)^2 \rightarrow y = 4 \rightarrow (-2,4)$$

$$x = 2 \rightarrow y = x^2 \rightarrow y = (2)^2 \rightarrow y = 4 \rightarrow (2,4)$$



Example: Graph the function $y = |x|$

$$|x| = \begin{cases} x, & x \geq 0 \\ -x, & x < 0, \end{cases}$$

Sol:

$$\text{when } y = x$$

$$\text{we put } x = 0 \rightarrow y = x \rightarrow y = 0 \rightarrow (0,0)$$

$$x = 1 \rightarrow y = x \rightarrow y = 1 \rightarrow (1,1)$$

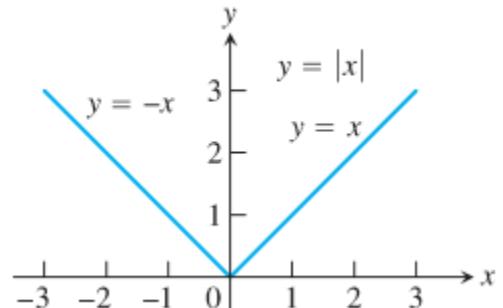
$$x = 2 \rightarrow y = x \rightarrow y = 2 \rightarrow (2,2)$$

$$\text{and when } y = -x$$

$$x = 0 \rightarrow y = -x \rightarrow y = 0 \rightarrow (0,0)$$

$$x = -1 \rightarrow y = -x \rightarrow y = 1 \rightarrow (-1,1)$$

$$x = -2 \rightarrow y = -x \rightarrow y = 2 \rightarrow (-2,2)$$



Example: Sketch the graph for the function

$$f(x) = \begin{cases} -x, & x < 0 \\ x^2, & 0 \leq x \leq 1 \\ 1, & x > 1 \end{cases}$$

Sol:

when $y = -x$

we put $x = 0 \rightarrow y = -x \rightarrow y = 0 \rightarrow (0,0)$

$x = -1 \rightarrow y = -x \rightarrow y = 1 \rightarrow (-1,1)$

$x = -2 \rightarrow y = -x \rightarrow y = 1 \rightarrow (-2,2)$

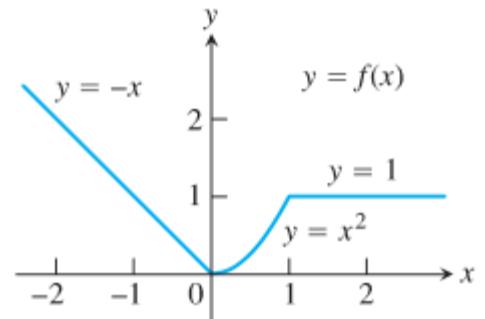
when $y = x^2$

$x = 0 \rightarrow y = x^2 \rightarrow y = 0 \rightarrow (0,0)$

and when $y = 1$

$x = 1 \rightarrow y = 1 \rightarrow (1,1)$

$x = 2 \rightarrow y = 1 \rightarrow (2,1)$



Even Functions and Odd Functions (Symmetry):

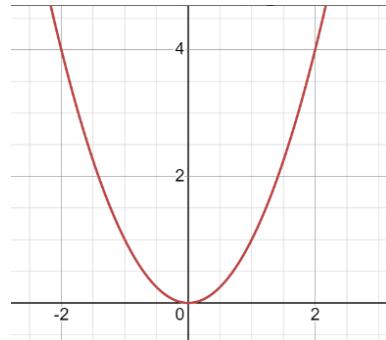
The graphs of even and odd functions have characteristic symmetry properties.

Definition: A function $y = f(x)$ is an:

-Even function of x if $f(-x) = f(x)$

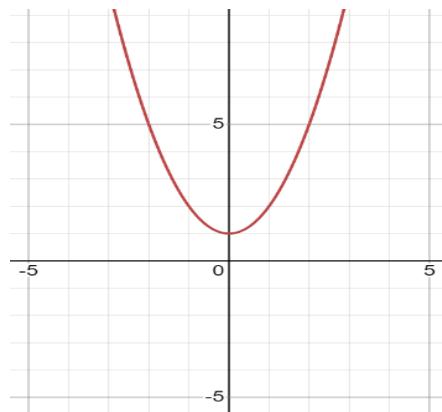
-Odd function of x if $f(-x) = -f(x)$

Example: $f(x) = x^2$ Even function: $(-x)^2 = x^2$ for all x ;



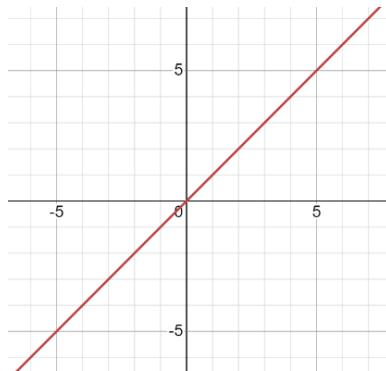
symmetry about $y - \text{axis}$.

$f(x) = x^2 + 1$ Even function: $(-x)^2 + 1 = x^2 + 1$ for all x ;



symmetry about $y - \text{axis}$.

$f(x) = x$ Odd function $(-x) = -x$ for all x ;



symmetry about the origin.

$$f(x) = x + 1 \quad \text{Not odd: } f(-x) = -x + 1,$$

but $-f(x) = -x - 1$. The two are not equal.

Not even: $(-x) + 1 \neq x + 1$ for all $x \neq 0$

Limit and Continuity:

When $f(x)$ close to the number L as x close to the number a , we write

$$f(x) \rightarrow L \text{ as } x \rightarrow a \quad \text{means: } \lim_{x \rightarrow a} f(x) = L$$

Example: Let $f(x) = 2x + 5$ evaluate $f(x)$ at $x = 1$

Sol:

$$\lim_{x \rightarrow a} f(x) = \lim_{x \rightarrow 1} (2x + 5) = 2 * 1 + 5 = 7$$

Example: If $f(x) = \frac{x^2 - 3x + 2}{x - 2}$, $x \neq 2$; find $\lim_{x \rightarrow 2} f(x)$.

Sol:

$$\lim_{x \rightarrow 2} f(x) = \lim_{x \rightarrow 2} \left(\frac{x^2 - 3x + 2}{x - 2} \right) = \lim_{x \rightarrow 2} \frac{(x-2)(x-1)}{x-2} = \lim_{x \rightarrow 2} (x-1) = 2 - 1 = 1$$

Example: Evaluate the following limits if they exist.

$$1) \lim_{x \rightarrow -1} \frac{\sqrt{2+x}-1}{x+1}, x \neq -1, x \neq -2$$

Sol:

$$\lim_{x \rightarrow -1} \frac{\sqrt{2+x}-1}{x+1} * \frac{\sqrt{2+x}+1}{\sqrt{2+x}+1} = \lim_{x \rightarrow -1} \frac{2+x-1}{x+1(\sqrt{2+x}+1)}$$

$$\lim_{x \rightarrow -1} \frac{x+1}{x+1(\sqrt{2+x}+1)} = \lim_{x \rightarrow -1} \frac{1}{(\sqrt{2+x}+1)} = \frac{1}{(\sqrt{2-1}+1)} = \frac{1}{2}$$

$$2) \lim_{x \rightarrow 2} \frac{2-x}{2-\sqrt{2x}} , x \neq 2, x \geq 0$$

Sol:

$$\lim_{x \rightarrow 2} \frac{2-x}{2-\sqrt{2x}} * \frac{2+\sqrt{2x}}{2+\sqrt{2x}} = \lim_{x \rightarrow 2} \frac{2-x(2+\sqrt{2x})}{4-2x}$$

$$\lim_{x \rightarrow 2} \frac{2-x(2+\sqrt{2x})}{2(2-x)} = \lim_{x \rightarrow 2} \frac{(2+\sqrt{2x})}{2} = \frac{(2+\sqrt{2*2})}{2} = 2$$

The Limit Laws:

If L, M, C, and k are real numbers and $\lim_{x \rightarrow c} f(x) = L$ and $\lim_{x \rightarrow c} g(x) = M$

1. Sum Rule: $\lim_{x \rightarrow c} (f(x) + g(x)) = L + M$

2. Difference Rule: $\lim_{x \rightarrow c} (f(x) - g(x)) = L - M$

3. Constant Multiple Rule: $\lim_{x \rightarrow c} k \cdot f(x) = k \cdot L$

4. Product Rule: $\lim_{x \rightarrow c} (f(x) \cdot g(x)) = L \cdot M$

5. Quotient Rule: $\lim_{x \rightarrow c} \frac{f(x)}{g(x)} = \frac{L}{M} , M \neq 0$

6. Power Rule: $\lim_{x \rightarrow c} [f(x)]^n = L^n$

7. Root Rule: $\lim_{x \rightarrow c} \sqrt[n]{f(x)} = \sqrt[n]{L}$, n is a positive integer.

Example: Evaluate the following limits

$$1) \lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1}, x \neq 1$$

Sol:

$$\lim_{x \rightarrow 1} \frac{x^3 - 1}{x - 1} = \lim_{x \rightarrow 1} \frac{(x - 1)(x^2 + x + 1)}{x - 1} = \lim_{x \rightarrow 1} x^2 + x + 1 = 1^2 + 1 + 1 = 3$$

$$2) \lim_{h \rightarrow 0} \frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right), h \neq 0$$

Sol:

$$\lim_{h \rightarrow 0} \left[\frac{1}{h} \left(\frac{1}{x+h} - \frac{1}{x} \right) \right] = \lim_{h \rightarrow 0} \left[\frac{1}{h} \left(\frac{x - x - h}{x(x+h)} \right) \right] = \lim_{h \rightarrow 0} \left[\frac{1}{h} \left(\frac{-h}{x(x+h)} \right) \right]$$

$$\lim_{h \rightarrow 0} \frac{-1}{x(x+h)} = -\frac{1}{x(x+0)} = -\frac{1}{x^2}$$

$$3) \lim_{x \rightarrow c} \frac{x^4 + x^2 - 1}{x^2 + 5} = \frac{\lim_{x \rightarrow c} (x^4 + x^2 - 1)}{\lim_{x \rightarrow c} (x^2 + 5)} = \frac{c^4 + c^2 - 1}{c^2 + 5}$$

$$4) \lim_{x \rightarrow -2} \sqrt{4x^2 - 3} = \sqrt{\lim_{x \rightarrow -2} (4x^2 - 3)} = \sqrt{4 * (-2)^2 - 3} = \sqrt{16 - 3} = \sqrt{13}$$

Limits of infinity:

We note when the limit of a function $f(x)$ exist and x approach at infinity, we write:

$$\lim_{x \rightarrow \infty} f(x) = L \quad \text{For positive values of } x.$$

$$\lim_{x \rightarrow -\infty} f(x) = L \quad \text{For negative values of } x.$$

Some obvious limits:

$$1- \text{ If } k \text{ is constant, then } \lim_{x \rightarrow \infty} k = k$$

$$2- \lim_{x \rightarrow \infty} \frac{1}{x} = 0, \lim_{x \rightarrow -\infty} \frac{1}{x} = \frac{1}{\infty} = 0$$

$$3- \lim_{x \rightarrow 0} \frac{1}{x} = \infty$$

Example: Find the following limits

$$1- \lim_{x \rightarrow \infty} \frac{x}{2x+3} = \lim_{x \rightarrow \infty} \frac{1}{2+\frac{3}{x}} = \frac{1}{2+0} = \frac{1}{2}$$

$$2- \lim_{x \rightarrow \infty} \frac{2x^2+3x+5}{5x^2-4x+1} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{3x}{x^2} + \frac{5}{x^2}}{\frac{5x^2}{x^2} - \frac{4x}{x^2} + \frac{1}{x^2}} = \frac{2}{5}$$

$$3- \lim_{x \rightarrow \infty} \frac{2x^2+1}{3x^2-2x^2+5x-2} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{1}{x^2}}{3 - \frac{2}{x} + \frac{5}{x^2} - \frac{2}{x^3}} = 0$$

$$4- \lim_{x \rightarrow \infty} \frac{2x^2+1}{3x^2-2x^2+5x-2} = \lim_{x \rightarrow \infty} \frac{\frac{2x^2}{x^2} + \frac{1}{x^2}}{3 - \frac{2}{x} + \frac{5}{x^2} - \frac{2}{x^3}} = 0$$

$$5- \lim_{x \rightarrow \infty} [(\sqrt{x^2+1}) - x * \frac{(\sqrt{x^2+1})+x}{(\sqrt{x^2+1})+x}] = \lim_{x \rightarrow \infty} \frac{x^2+1-x^2}{(\sqrt{x^2+1})+x}$$

$$\lim_{x \rightarrow \infty} \frac{1}{(\sqrt{x^2+1})+x} = \frac{\frac{1}{x^2}}{\frac{(\sqrt{x^2+1})+x}{x}} = 0$$

Continuous Function:

A function $f(x)$ is continuous at an interior point $x = c$ of its domain if and only if it meets the following three conditions.

1- $f(c)$ is exists.

2- $\lim_{x \rightarrow c} f(x) = \text{exists.}$

3- $\lim_{x \rightarrow c} f(x) = c$

Example:

1) $f(x) = \frac{1}{x}$ is not continuous for all except $x = 0$

2) $f(x) = \frac{x+3}{(x-5)(x+2)}$ is discontinuous at $x = 5$ and $x = -2$

3) $f(x) = \frac{\sin x}{x}$ is discontinuous at $x = 0$

4) $f(x) = \frac{x^2+x-6}{x^2-4}$ is discontinuous at $x = \pm 2$

HOMEWORK

1- Solve the inequalities

a) $|x + 3| \leq 6$ b) $7 > |2x + 3|$

2-Graph the following functions

a) $y = x + 1$, $-1 \leq x \leq 1$

b) $x^2 + y = 2$

c) $y = 1 - x^2 \quad 0 \leq x \leq 3$

3-Find the limits

a) $\lim_{x \rightarrow -2} 5$

$$b) \lim_{m \rightarrow 0} \frac{\sqrt{3m+4}-2}{m-1}$$

$$\text{c)} \lim_{y \rightarrow 1} \frac{\sqrt{y+1} - \sqrt{2y}}{y^2 - y}$$

$$d) \lim_{x \rightarrow \infty} \frac{2x^4 - x^2 + 3}{x + x^4}$$

4- Write an equation for each line described by:

a- Passes through $(-1,1)$ with slope -1

b- Passes through $(2,5)$ and $(-1,2)$.

5- Find the slope of the curve and the tangent line of $y = 2x - x^2 - 3$ at $(0, 2)$.

Chapter Three

Differentiation

Definition:

The derivative of a function $f(x)$ with respect to the variable x is the function $\dot{f}(x)$ whose value at x is:

$$\dot{f}(x_0) = \lim_{h \rightarrow 0} \frac{f(x_0 + h) - f(x_0)}{h}$$

Provided this limit exists.

Differentiation Rules:

1-Derivative of a constant function.

If $y = f(x) = c$ where c is constant then,

$$\frac{dy}{dx} = \dot{f}(x) = 0$$

Example: $f(x) = 1$ then $\frac{dy}{dx} = \dot{f}(x) = 0$

Example: $f(x) = 25$ then $\frac{dy}{dx} = \dot{f}(x) = 0$

2-Power rule for positive integers:

If n is a positive integer, and $y = f(x) = x^n$; then

$$f'(x) = \frac{dy}{dx} = n x^{n-1}$$

Example: $y = f(x) = x^3$ then $f'(x) = \frac{dy}{dx} = 3 x^2$

Example: $y = f(x) = x^8$ then $f'(x) = \frac{dy}{dx} = 8 x^7$

3-Derivative constant multiple rule:

If u is a differentiable function of x $f(x) = c u(x)$, and c is a constant, then

$$\frac{dy}{dx} = f'(x) = c u'(x)$$

Example: $f(x) = 5x^7$ then $\frac{dy}{dx} = 5 * 7 x^6 = 35 x^6$

Example: $f(x) = 2x^4$ then $\frac{dy}{dx} = 2 * 4 x^3 = 8 x^3$

4-Derivative Product rule:

If u and v are differentiable at x, then so is their product uv , and

$$\frac{d}{dx}(uv) = u \frac{dv}{dx} + v \frac{du}{dx}$$

Example: Find the derivative of $y = (x^2) \cdot (x^3)$

Sol:

$$\frac{dy}{dx} = (x^2) \cdot 3x^2 + (x^3) \cdot (2x)$$

Example: Find the derivative of $y = (x^2 + 8x)(x^3 - 1)$

Sol:

$$\frac{dy}{dx} = (x^2 + 8x) \cdot 3x^2 + (x^3 - 1) \cdot (2x + 8)$$

5-Derivative quotient rule:

If u and v are differentiable at x and if $v(x) \neq 0$, then the quotient $\frac{u}{v}$ is differentiable at x , and

$$\frac{d}{dx}\left(\frac{u}{v}\right) = \frac{v \frac{du}{dx} - u \frac{dv}{dx}}{v^2}$$

Example: Find the derivative of $y = \frac{x^2}{x^3}$

Sol:

$$\frac{dy}{dx} = \frac{(x^3) \cdot (2x) - (x^2) \cdot 3x^2}{(x^3)^2} = \frac{2x^4 - 3x^4}{x^6}$$

Example: Find the derivative of $y = \frac{t-t^2}{t+4}$

Sol:

$$\frac{d}{dx}\left(\frac{t-t^2}{t+4}\right) = \frac{(t+4) \cdot (1-2t) - (t-t^2) \cdot 1}{(t+4)^2} = \frac{(t+4)(1-2t) - (t-t^2)}{(t+4)^2}$$

Second and Higher- Order Derivatives:

If $f(x)$ is a given function then

$$\frac{dy}{dx} = f(x) \text{ is first derivative of } y.$$

$$\frac{d^2y}{dx^2} = \hat{f}(x) \text{ is second derivative of } y.$$

$$\frac{d^3y}{dx^3} = \hat{\hat{f}}(x) \text{ is third derivative of } y. \text{ And so on...}$$

Then, in general:

$$\frac{d^n y}{dx^n} = f^n(x) = y^n$$

Example: If $y = (x^2 + 2x + 3)^2$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}$

Sol:

$$\frac{dy}{dx} = \dot{y} = 2 \cdot (x^2 + 2x + 3)(2x + 2)$$

$$\frac{d^2y}{dx^2} = 2 [(x^2 + 2x + 3) \cdot 2 + (2x + 2) \cdot (2x + 2)]$$

$$\frac{d^2y}{dx^2} = 4(x^2 + 2x + 3) + 2(2x + 2)^2$$

Example: If $y = 2x^3 - 4x^2 + 6x - 5$, find $\frac{dy}{dx}$ and $\frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}, \frac{d^4y}{dx^4}$

Sol:

$$\frac{dy}{dx} = 6x^2 - 8x + 6$$

$$\frac{d^2y}{dx^2} = 12x - 8$$

$$\frac{d^3y}{dx^3} = 12$$

$$\frac{d^4y}{dx^4} = 0$$

Chain rule:

If y is a function of x , say $y = f(x)$, and x is a function of t , say $x = g(t)$ then y is a function of t :

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

This formula is called chain rule.

Example: If $y = x^3 - x^2 + 5$ and $x = 2t^2 + t$, find $\frac{dy}{dt}$ at $t = 1$.

Sol:

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt} = (3x^2 - 2x)(4t + 1)$$

$$\text{at } t = 1 \rightarrow x = (2)1^2 + 1 = 3$$

$$\frac{dy}{dt} = (3 * 3^2 - 2 * 3)(4 * 1 + 1) = 105$$

Implicit Differentiation:

Most of the functions we have dealt with so far have been described by an equation of the form $y = f(x)$ that expresses y explicitly in terms of the variable x . We have learned rules for differentiating functions defined in this way. Another situation occurs when we encounter equations like

$$y^2 - x^2 = 0, y^3 + 8x = 3, \quad yx + 4y - 6 = 0$$

Definition:

1. Differentiate both sides of the equation with respect to x , treating y as a differentiable function of x .
2. Collect the terms with dy / dx on one side of the equation and solve for dy / dx .

Example: Find $\frac{dy}{dx}$ for the equation $y^2 + x^3 - 9xy = 0$

Sol:

$$2y\frac{dy}{dx} + 3x^2 - \left(9x\frac{dy}{dx} + 9y\right) = 0 \rightarrow 2y\frac{dy}{dx} + 3x^2 - 9x\frac{dy}{dx} - 9y = 0$$

$$2y\frac{dy}{dx} - 9x\frac{dy}{dx} = 9y - 3x^2 \rightarrow \frac{dy}{dx}(2y - 9x) = 9y - 3x^2$$

$$\frac{dy}{dx} = \frac{9y - 3x^2}{2y - 9x}$$

Example: Use implicit differentiation to find dy / dx for the equations

$$1- y^2 = \frac{x-1}{x+1}$$

$$2- x\cos(2x + 3) = y\sin x$$

Sol:

$$1- y^2 = \frac{x-1}{x+1}$$

Sol:

$$2y \frac{dy}{dx} = \frac{(x+1) \cdot 1 - (x-1) \cdot 1}{(x-1)^2} = \frac{x+1-x+1}{(x-1)^2} = \frac{2}{(x-1)^2}$$

$$\frac{dy}{dx} = \frac{2}{2y(x-1)^2} = \frac{1}{y(x-1)^2}$$

$$2- x \cos(2x+3) = y \sin x$$

Sol:

$$-x \sin(2x+3) \cdot 2 + \cos(2x+3) \cdot 1 = y \cos x + \frac{dy}{dx} \sin x$$

$$-2x \sin(2x+3) + \cos(2x+3) = y \cos x + \frac{dy}{dx} \sin x$$

$$-2x \sin(2x+3) + \cos(2x+3) - y \cos x = \frac{dy}{dx} \sin x$$

$$\frac{dy}{dx} = \frac{-2x \sin(2x+3) + \cos(2x+3) - y \cos x}{\sin x}$$

Application of Differentiation:

a) Partial Derivatives:

The partial derivative of $f(x, y)$ with respect to x at the point (x_0, y_0) is:

$$\left. \frac{\partial f}{\partial x} \right|_{(x_0, y_0)} = \frac{d}{dx} f(x, y_0) = f_x$$

The partial derivative of $f(x, y)$ with respect to y at the point (x_0, y_0) is:

$$\left. \frac{\partial f}{\partial y} \right|_{(x^*, y^*)} = \frac{d}{dy} f(x^*, y) = f_y$$

Example: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ at the point $(2, -1)$ if

$$f(x, y) = x^2 + 3xy + y - 1$$

Sol:

To find $\frac{\partial f}{\partial x}$, we treat y as a constant and differentiate with respect to x ,

$$\frac{\partial f}{\partial x} = 2x + 3y + 0 - 0 = 2x + 3y$$

The values of $\frac{\partial f}{\partial x}$ at the point $(2, -1)$ is: $2 * 2 + 3 * -1 = 1$

To find $\frac{\partial f}{\partial y}$, we treat x as a constant and differentiate with respect to y ,

$$\frac{\partial f}{\partial y} = 0 + 3x + 1 - 0 = 3x + 1$$

The values of $\frac{\partial f}{\partial y}$ at the point $(2, -1)$ is: $3 * 2 + 1 = 7$

Example: Find the values of $\frac{\partial f}{\partial y}$ if $f(x, y) = y \sin(xy)$

Sol:

$$\frac{\partial f}{\partial y} = y * \cos(xy) * x + \sin(xy) * 1 = xy \cos(xy) + \sin(xy)$$

Example: Find the values of $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ if $f(x, y) = \frac{2y}{y+\cos x}$

Sol:

$$\frac{\partial f}{\partial x} = \frac{(y + \cos x) * 0 - [2y(0 - \sin x)]}{(y + \cos x)^2} = \frac{2y \sin x}{(y + \cos x)^2}$$

$$\frac{\partial f}{\partial y} = \frac{(y + \cos x) * 2 - [2y(1+0)]}{(y + \cos x)^2} = \frac{2(y + \cos x) - 2y}{(y + \cos x)^2}$$

b) Indeterminate Forms and Hôpital's Rule:

Suppose that $f(a) = g(a) = 0$, that $\hat{f}(a), \hat{g}(a)$ exist, and that $\hat{g}(a) \neq 0$

Then;

$$\lim_{x \rightarrow 0} \frac{f(x)}{g(x)} = \frac{\hat{f}(a)}{\hat{g}(a)}$$

Example: Using Hospital's Rule and find the following:

$$1) \lim_{x \rightarrow 0} \frac{3x - \sin x}{x} = \frac{3 - \cos x}{1} \Big|_{x=0} = 2$$

$$2) \lim_{x \rightarrow 0} \frac{\sqrt{x+1} - 1}{x} = \frac{\frac{1}{2\sqrt{x+1}}}{1} \Big|_{x=0} = \frac{1}{2}$$

$$3) \lim_{x \rightarrow 2} \frac{x-2}{x^2-4} = \frac{1}{2x} \Big|_{x=2} = \frac{1}{4}$$

$$4) \lim_{x \rightarrow 1} \frac{x^3-1}{4x^3-x-3} = \frac{3x^2}{12x^2-1} \Big|_{x=1} = \frac{3}{11}$$

Transcendental Function:

1-Trigonometric function:

- 1- If $y = f(x) = \sin x$ then $\frac{dy}{dx} = \cos x$
- 2- If $y = f(x) = \cos x$ then $\frac{dy}{dx} = -\sin x$
- 3- If $y = f(x) = \tan x$ then $\frac{dy}{dx} = \sec^2 x$
- 4- If $y = f(x) = \cot x$ then $\frac{dy}{dx} = -\csc^2 x$
- 5- If $y = f(x) = \sec x$ then $\frac{dy}{dx} = \sec x \tan x$
- 6- If $y = f(x) = \csc x$ then $\frac{dy}{dx} = -\csc x \cot x$

Now if u is a function of x then:

- 1- If $y = \sin u$ then $\frac{dy}{du} = \cos u \frac{du}{dx}$
- 2- If $y = \cos u$ then $\frac{dy}{du} = -\sin u \frac{du}{dx}$
- 3- If $y = \tan u$ then $\frac{dy}{du} = \sec^2 u \frac{du}{dx}$
- 4- If $y = \cot u$ then $\frac{dy}{du} = -\csc^2 u \frac{du}{dx}$

- 5- If $y = \sec u$ then $\frac{dy}{du} = \sec u \tan u \frac{du}{dx}$
- 6- If $y = \csc u$ then $\frac{dy}{du} = -\csc u \cot u \frac{du}{dx}$

Example: Find $\frac{dy}{dx}$ of the following functions

$$(1) y = \sin(x^2 + 2x - 5)$$

Sol:

$$\frac{dy}{dx} = \cos(x^2 + 2x - 5) \cdot 2x + 2 = (2x + 2) \cos(x^2 + 2x - 5)$$

$$(2) y = \tan(2x) \cos(x^2 + 1)$$

Sol:

$$\frac{dy}{dx} = -\tan(2x) \sin(x^2 + 1) \cdot 2x + \cos(x^2 + 1) \sec^2(2x) \cdot 2$$

$$\frac{dy}{dx} = -2x \tan(2x) \sin(x^2 + 1) + 2 \cos(x^2 + 1) \sec^2(2x)$$

$$(3) y = \sin^2\left(x^2 + \frac{1}{x^2}\right)$$

$$\frac{dy}{dx} = 2 \sin\left(x^2 + \frac{1}{x^2}\right) \cdot \cos\left(x^2 + \frac{1}{x^2}\right) \cdot \left(2x - \frac{2}{x^3}\right)$$

$$(4) y = \tan^{-3}(3x^2 + \sec^2 2x)$$

$$\begin{aligned} \frac{dy}{dx} = & -3 \tan^{-4}(3x^2 + \sec^2 2x) \cdot \sec^2(3x^2 + \sec^2 2x) \cdot (6x + \\ & 2 \sec 2x \cdot \sec 2x \tan 2x \cdot 2) \end{aligned}$$

$$(5) y = \frac{\sec[\sin(2x+1)]}{\tan(x^3+1)}$$

$$\frac{dy}{dx} = \frac{\tan(x^3+1) \sec[\sin(2x+1)] \tan[\sin(2x+1)] \cdot \cos(2x+1) \cdot 2 - \sec[\sin(2x+1)] \sec^2(x^3+1) \cdot 3x^2}{[\tan(x^3+1)]^2}$$

Example: If $y = \tan^{-3}(\sin 2x)$, find $\frac{dy}{dx}$

Sol:

$$\frac{dy}{dx} = -3 \tan^{-4}(\sin 2x) \cdot \sec^2(\sin 2x) \cdot \cos 2x \cdot 2 = -\frac{6 \sec^2(\sin 2x) \cos 2x}{\tan^4(\sin 2x)}$$

Inverse of trigonometric function:

Definition: If $y = f(x)$ and $\frac{dy}{dx} = y$; then:

- (1) $y = \sin^{-1} x \rightarrow x = \sin y$
- (2) $y = \cos^{-1} x \rightarrow x = \cos y$
- (3) $y = \tan^{-1} x \rightarrow x = \tan y$
- (4) $y = \cot^{-1} x \rightarrow x = \cot y$
- (5) $y = \sec^{-1} x \rightarrow x = \sec y$
- (6) $y = \csc^{-1} x \rightarrow x = \csc y$

Derivative of the inverse trigonometric function:

- (1) $y = \sin^{-1} x \rightarrow \dot{y} = \frac{1}{\sqrt{1-x^2}}$
- (2) $y = \cos^{-1} x \rightarrow \dot{y} = \frac{-1}{\sqrt{1-x^2}}$
- (3) $y = \tan^{-1} x \rightarrow \dot{y} = \frac{1}{1+x^2}$
- (4) $y = \cot^{-1} x \rightarrow \dot{y} = \frac{-1}{1+x^2}$
- (5) $y = \sec^{-1} x \rightarrow \dot{y} = \frac{1}{x\sqrt{x^2-1}}$
- (6) $y = \csc^{-1} x \rightarrow \dot{y} = \frac{-1}{x\sqrt{x^2-1}}$

Example: Find $\frac{dy}{dx}$ of the following functions:

Sol:

$$1) y = \sin^{-1}(x^2 + 3x - 1)$$

$$\frac{dy}{dx} = \frac{2x+3}{\sqrt{1-(x^2+3x-1)^2}}$$

$$2) y = x^2 \tan^{-1}(\sqrt{x})$$

$$\frac{dy}{dx} = x^2 \cdot \frac{1}{1+x^2} \cdot \frac{1}{2\sqrt{x}} + 2x \tan^{-1}(\sqrt{x})$$

$$3) y = \cos^{-1}(x^2 + \tan^{-1} 3x)$$

$$\frac{dy}{dx} = \frac{-1}{\sqrt{1-(x^2 + \tan^{-1} 3x)^2}} \cdot \left(2x + \frac{3}{1+9x^2} \right) = \frac{-\left(2x + \frac{3}{1+9x^2} \right)}{\sqrt{1-(x^2 + \tan^{-1} 3x)^2}}$$

4) $y = \sin^2(\sec^{-1} 2x) \cot^{-1}\left(\frac{1}{x}\right)$

$$\frac{dy}{dx} = \sin^2(\sec^{-1} 2x) \cdot \frac{-1}{1+\left(\frac{1}{x}\right)^2} \cdot \frac{-1}{x^2} + \cot^{-1}\left(\frac{1}{x}\right) \cdot 2 \sin(\sec^{-1} 2x) \cdot \cos(\sec^{-1} 2x) \cdot \frac{2}{2x\sqrt{4x^2-1}}$$

Example: If $y = \sin^{-1}\left(\frac{x-1}{x+1}\right)$ find $\frac{dy}{dx}$

Sol:

$$\frac{dy}{dx} = \frac{1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2}} \cdot \frac{(x+1)\cdot 1 - (x-1)\cdot 1}{(x+1)^2} = \frac{x+1-x+1}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2(x+1)^2}} = \frac{2}{\sqrt{1-\left(\frac{x-1}{x+1}\right)^2(x+1)^2}}$$

Example: If $y = \sin^{-1} t$, $x = \cos^{-1} t$, find $\frac{dy}{dx}, \frac{d^2y}{dx^2}$

Sol:

$$y = \sin^{-1} t \rightarrow \frac{dy}{dt} = \frac{1}{\sqrt{1-t^2}}$$

$$x = \cos^{-1} t \rightarrow \frac{dx}{dt} = \frac{-1}{\sqrt{1-t^2}}$$

$$\frac{dy}{dx} = \frac{dy/dt}{dx/dt} = \frac{\frac{1}{\sqrt{1-t^2}}}{\frac{-1}{\sqrt{1-t^2}}} = -1$$

$$\frac{d^2y}{dx^2} = 0$$

The Logarithmic function:

The logarithmic was discovered by Noble man John Napier (1550-1617)

$$y = f(x) = \log_b x \rightarrow x = b^y$$

Where y is the logarithm, x is the number, b is the base.

If $b = 10$, we write $y = \log x$ or $y = \log x$

If $b = e = 2.7183$, we write

$$y = \log_e x \quad \text{or} \quad \log x = y \rightarrow y = \ln x$$

Where (\ln) is read natural logarithm.

Relation between the logarithm and the natural logarithm:

Let $y = \log_b x \rightarrow x = b^y \rightarrow \ln x = \ln b^y = y \ln b$

$$\therefore y = \frac{\ln x}{\ln b} \quad \text{so} \quad \log_b x = \frac{\ln x}{\ln b}$$

Properties:

- $\ln(a \cdot b) = \ln a + \ln b$
- $\ln\left(\frac{a}{b}\right) = \ln a - \ln b$
- $\ln 1 = 0$, $\ln a^r = r \ln a$

Derivative of the natural logarithm:

If $y = f(x) = \ln x$ then: $\frac{dy}{dx} = f'(x) = \frac{1}{x}$

Example: Find $\frac{dy}{dx}$ of the following functions:

$$1- y = \ln(x^3 + 2x^2 - 1)$$

Sol:

$$\frac{dy}{dx} = \frac{1}{x^3 + 2x^2 - 1} \cdot (3x^2 + 4x)$$

$$2- y = \ln(x^{-2} + \sin^2 3x)$$

Sol:

$$\frac{dy}{dx} = \frac{1}{x^{-2} + \sin^2 3x} \cdot -2x^{-3} + 2 \sin(3x) \cos(3x) \cdot 3$$

$$3- y = \sin^{-1}(\ln x) \cdot \ln(\sin^{-1} 3x)$$

Sol:

$$\frac{dy}{dx} = \sin^{-1}(\ln x) \cdot \frac{1}{\sin^{-1} 3x} \cdot \frac{3}{\sqrt{1 - (3x)^2}} + \ln(\sin^{-1} 3x) \cdot \frac{1}{\sqrt{1 - (\ln x)^2}} \cdot \frac{1}{x}$$

$$4- y = \ln[\ln(\sec^2 2x + x \sin^{-1} x)]$$

Sol:

$$\begin{aligned} \frac{dy}{dx} &= \frac{1}{\ln(\sec^2 2x + x \sin^{-1} x)} \cdot \frac{1}{\sec^2 2x + x \sin^{-1} x} \cdot 2 \sec(2x) (\sec(2x) \tan(2x) \cdot 2) \\ &\quad + \frac{x}{\sqrt{1 - x^2}} + \sin^{-1} x \end{aligned}$$

Example: Find $\frac{dy}{dx}$ for the following:

1) $y = x^{\cos x}$

Sol: Take \ln for both sides

$$y = x^{\cos x} \rightarrow \ln y = \ln x^{\cos x}$$

$$\ln y = \cos x \ln x$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{\cos x}{x} - \sin x \ln x$$

$$\frac{dy}{dx} = y \left[\frac{\cos x}{x} - \sin x \ln x \right] = x^{\cos x} \left[\frac{\cos x}{x} - \sin x \ln x \right]$$

2) $y = (\ln x)^x$

Sol:

$$\ln y = \ln [(\ln x)^x] = \ln [x \ln x]$$

$$\frac{1}{y} \frac{dy}{dx} = \frac{1}{x \ln x} \cdot (x \cdot \frac{1}{x} + \ln x) = \frac{\ln x + 1}{x \ln x}$$

$$\therefore \frac{dy}{dx} = y \left[\frac{\ln x + 1}{x \ln x} \right] = (\ln x)^x \left[\frac{\ln x + 1}{x \ln x} \right]$$

Example: Solve for x if $2^x = 4^{x-1}$

Sol:

Take \ln for both sides

$$\ln(2^x) = \ln(4^{x-1})$$

$$x \ln 2 = (x - 1) \ln 4 = x \ln 4 - \ln 4$$

$$x \ln 2 - x \ln 4 = -\ln 4$$

$$x(\ln 2 - \ln 4) = -\ln 4 \rightarrow \therefore x = \frac{-\ln 4}{\ln 2 - \ln 4}$$

The Exponential function:

Definition: the exponential function is defined as an inverse of natural logarithm, and denoted by: **exp** or **e**, that is:

For $-\infty < x < \infty$ we define $y = f(x) = e^x$

then $x = \ln y, 0 < y < \infty$

Properties:

$$1- e = 2.7183$$

$$2- e^{x+y} = e^x \cdot e^y$$

$$3- e^{x-y} = e^x \cdot e^{-y}$$

$$4- e^{\ln x} = x$$

$$5- \ln e^x = x$$

Example: Simplify the following expressions.

$$1- e^{\ln 2} = 2$$

$$2- \ln e^{\sin x} = \sin x$$

$$3- e^{\ln(x^2+1)} = x^2 + 1$$

$$4- \ln e^{-1.3} = -1.3$$

$$5- \ln \frac{e^{2x}}{5} = \ln e^{2x} - \ln 5 = 2x - \ln 5$$

$$6- e^{\ln 2 + 3 \ln x} = e^{\ln 2} \cdot e^{3 \ln x} = 2 \cdot e^{\ln x^3} = 2x^3$$

$$7- e^{2x + \ln x} = e^{2x} \cdot e^{\ln x} = x e^{2x}$$

Example: Solve for y if: $\ln(y-1) - \ln y = 2x$

Sol:

$$\ln(y-1) - \ln y = 2x \rightarrow \ln \frac{y-1}{y} = 2x \quad (\text{Take } \exp \text{ for both sides})$$

$$e^{\ln \frac{y-1}{y}} = e^{2x} \rightarrow \frac{y-1}{y} = e^{2x}$$

$$y-1 = y e^{2x} \rightarrow y - y e^{2x} = 1$$

$$y(1 - e^{2x}) = 1 \rightarrow y = \frac{1}{1 - e^{2x}}$$

Derivative of the exponential function:

$$\text{If } y = e^x \text{ then } \frac{dy}{dx} = e^x$$

Now, if $u = u(x)$ then $y = e^u$

$$\frac{du}{dx} = e^u \cdot \frac{du}{dx}$$

Example: Find $\frac{dy}{dx}$ of the following functions.

$$1- \ y = e^{x^2 + \sin 2x}$$

$$\frac{dy}{dx} = e^{x^2 + \sin 2x} \cdot (2x + 2 \cos 2x)$$

$$2- \ y = e^{(\tan^{-1} 2x) + \ln x}$$

$$\frac{dy}{dx} = e^{(\tan^{-1} 2x) + \ln x} \cdot \left(\frac{2x}{1 + 4x^2} + \frac{1}{x} \right)$$

$$3- \ y = \tan^{-1}(e^{2x})$$

$$\frac{dy}{dx} = \frac{1}{1+e^{4x^2}} \cdot e^{2x} \cdot 2 = \frac{2e^{2x}}{1+e^{4x^2}}$$

$$4- \ y = e^{\sec x} \cdot \sec e^x$$

$$\frac{dy}{dx} = e^{\sec x} \cdot (\sec e^x \tan e^x \cdot e^x) + \sec e^x \cdot e^{\sec x} \cdot (\sec x \tan x)$$

The Function a^x :

Definition: for $a > 0$, we define $a^x = e^{x \ln a}$

If $y = a^x$ then

$$\frac{dy}{dx} = a^x \ln a$$

Now if $u = u(x)$ then $y = a^x$

$$\frac{dy}{dx} = a^x \cdot \ln a \frac{du}{dx}$$

Example: Find $\frac{dy}{dx}$ of the following functions.

$$1- \quad y = 2^{\sin^2 2x}$$

$$\frac{dy}{dx} = 2^{\sin^2 2x} \cdot \ln 2 (2 \sin 2x \cos 2x \cdot 2)$$

$$2- \quad y = 3^{\tan^{-1} 2x}$$

$$\frac{dy}{dx} = 3^{\tan^{-1} 2x} \cdot \ln 3 \frac{2}{1 + 4x^2}$$

2-Hyperbolic functions:

The hyperbolic functions are a special combinations of the functions e^x and e^{-x} .

Definitions:

$$1- \quad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$2- \quad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$3- \quad \tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

$$4- \quad \coth x = \frac{e^x + e^{-x}}{e^x - e^{-x}} = \frac{1}{\tanh x}$$

$$5- \quad \operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

$$6- \quad \operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}}$$

Some important relations and identities:

- 1) $\cosh^2 x - \sinh^2 x = 1$
 - 2) $\tanh^2 x + \coth^2 x = 1$
 - 3) $\coth^2 x - \operatorname{csch}^2 x = 1$
 - 4) $\sinh(-x) = -\sinh x$
 - 5) $\coth(-x) = \coth x$
 - 6) $\tanh(-x) = -\tanh x$
 - 7) $\sinh x \pm \cosh x = \pm e^{\pm x}$
 - 8) $\sinh(x \pm y) = \sinh y \cosh x \pm \sinh x \cosh y$
 - 9) $\cosh(x \pm y) = \cosh x \cosh y \pm \sinh x \sinh y$
 - 10) $\sinh 2x = 2 \sinh x \cosh x$
- 11) $\sinh^2 x = \frac{\cosh(2x)-1}{2}$
- 12) $\cosh^2 x = \frac{\cosh(2x)+1}{2}$

Derivative of hyperbolic functions:

- 1) $y = \sinh x \rightarrow \frac{dy}{dx} = \cosh x$
- 2) $y = \cosh x \rightarrow \frac{dy}{dx} = \sinh x$
- 3) $y = \tanh x \rightarrow \frac{dy}{dx} = \operatorname{sech}^2 x$
- 4) $y = \coth x \rightarrow \frac{dy}{dx} = -\operatorname{csch}^2 x$
- 5) $y = \operatorname{sech} x \rightarrow \frac{dy}{dx} = -\operatorname{sech} x \tanh x$
- 6) $y = \operatorname{csch} x \rightarrow \frac{dy}{dx} = -\operatorname{csch} x \coth x$

Example: Find $\frac{dy}{dx}$ of the following functions.

- 1) $y = \sinh(x^2 + 3 \sin x + \ln x)$

$$\frac{dy}{dx} = \cosh(x^2 + 3 \sin x + \ln x) \cdot (2x + 3 \cos x + \frac{1}{x})$$

$$2) y = \tanh^{-2}(e^{\tanh^{-1} 2x} + \sin e^{2x})$$

$$\begin{aligned}\frac{dy}{dx} = & -2 \tanh^{-3}(e^{\tanh^{-1} 2x} + \sin e^{2x}) \cdot \operatorname{sech}^2(e^{\tanh^{-1} 2x} \\ & + \sin e^{2x}) \cdot e^{\tanh^{-1} 2x} \cdot \left(\frac{2}{1+4x^2} \right) + \cos e^{2x} \cdot (e^{2x} \cdot 2)\end{aligned}$$

Inverse of hyperbolic function:

- 1) $y = \sinh^{-1} x \rightarrow x = \sinh y$
- 2) $y = \cosh^{-1} x \rightarrow x = \cosh y$
- 3) $y = \tanh^{-1} x \rightarrow x = \tanh y$
- 4) $y = \coth^{-1} x \rightarrow x = \coth y$
- 5) $y = \operatorname{sech}^{-1} x \rightarrow x = \operatorname{sech} y$
- 6) $y = \operatorname{csch}^{-1} x \rightarrow x = \operatorname{csch} y$

Derivative of inverse of hyperbolic function:

- 1) $y = \sinh^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{1+x^2}}$
- 2) $y = \cosh^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{\sqrt{x^2-1}}$
- 3) $y = \tanh^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$
- 4) $y = \coth^{-1} x \rightarrow \frac{dy}{dx} = \frac{1}{1-x^2}$
- 5) $y = \operatorname{sech}^{-1} x \rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1-x^2}}$
- 6) $y = \operatorname{csch}^{-1} x \rightarrow \frac{dy}{dx} = \frac{-1}{x\sqrt{1+x^2}}$

Example: Find $\frac{dy}{dx}$ to the following function: $y = \sinh^{-1}(x^2 + \sin^2 x)$

Sol:

$$\frac{dy}{dx} = \frac{2x+2 \sin x \cos x}{\sqrt{1+(x^2+\sin^2 x)^2}}$$

HOMEWORK

1) Find the first and second derivatives:

a- $y = z^6 - 7z^2 + 21z$

b- $y = x^3 - 4x$

c- $y = 3 \sin^2 x$

2) Simplify the expression:

a- $6^{\log(e^{\sin x})}$

b- $\log_2 5x$

c- $\log_e e^{x^3}$

d- $8^{\log 8x}$

3) Solve the equations:

a- $\ln e + 2^{-2 \log_2 x} = \log_3 9$

c- $4^{\log_4 x^2} = e^{\ln x} + 3^{\log_3 2}$

4) Find $\frac{dy}{dx}$:

a- $2e^{2y} = \ln x^2$

b- $y = (\ln x + x)^{\tan x}$

d- $y = \ln x^{e^x}$

Chapter Four

Integration

Indefinite Integrals:

Definition: The set of all antiderivatives of f is the indefinite integral of f with respect to x , denoted by:

$$\int f(x) dx = F(x) + c$$

The symbol \int is an integral sign. The function is f the integrand of the integral, and x is the variable of integration and c is the constant of integral.

Some integration formulas:

$$1) \int \frac{du}{dx} dx = u(x) + c$$

$$2) \int a u(x) dx = a \int u(x) dx, a \text{ is constant.}$$

$$3) \int [u_1(x) + u_2(x) + \dots] dx = \int u_1(x) dx + \int u_2(x) dx + \dots$$

$$4) \int u^n \frac{du}{dx} dx = \frac{u^{n+1}}{n+1} + c, n \neq -1$$

Example: Evaluate the integral $\int (4x^2 + 2x - 1) dx$

Sol:

$$\begin{aligned} \int (4x^2 + 2x - 1) dx &= 4 \int x^2 dx + 2 \int x dx - \int dx \\ &= 4 \frac{x^3}{3} + \frac{2x^2}{2} - x + c = \frac{4}{3}x^3 + x^2 - 5x + c \end{aligned}$$

Example: Evaluate the integral $\int (4x - x^2)^2 (4 - 2x) dx$

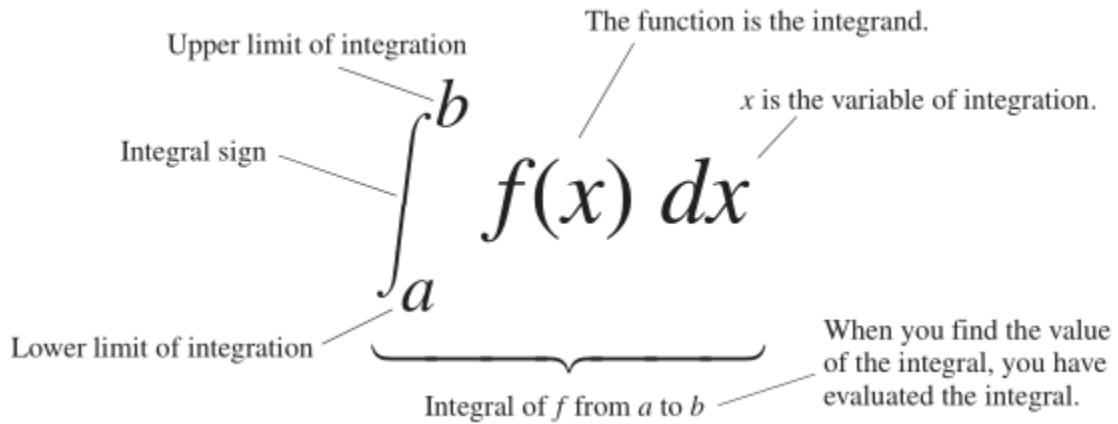
Sol:

$$\int (4x - x^2)^2 dx = \frac{(4x - x^2)^3}{3} + c = \frac{1}{3} (4x - x^2)^3 + c$$

Definite Integral:

The integral $\int_a^b f(x) dx$ is called the definite integral of $f(x)$ over the interval

$[a, b]$.



Properties of definite integrals:

If $f(x)$ is a continuous function on $[a, b]$ then:

$$1) \int_a^b f(x) dx = - \int_b^a f(x) dx$$

$$2) \int_a^a dx = 0$$

$$3) \int_a^b k f(x) dx = k \int_a^b f(x) dx, k = \text{constant}.$$

$$4) \int_a^b [f_1(x) + f_2(x) + f_3(x) + \dots] dx = \int_a^b f_1 dx + \int_a^b f_2 dx + \int_a^b f_3 dx + \dots$$

$$5) \int_a^b f(x) dx = \int_a^c f(x) dx + \int_c^b f(x) dx, \text{ for } c \in [a, b].$$

The fundamental theorem of integral calculus:

If $f(x)$ is continuous function on $[a, b]$ and $F(x)$ is any solution of $f(x)$ over $[a, b]$, then:

$$\int_a^b f(x) dx = F(x)|_a^b = F(b) - F(a)$$

Example: Evaluate $\int_{-3}^2 (6 - x - x^2) dx$

Sol:

$$\int_{-3}^2 (6 - x - x^2) dx = 6x - \frac{x^2}{2} - \frac{x^3}{3} \Big|_{-3}^2 = \left(12 - \frac{4}{2} - \frac{8}{3}\right) - \left(-18 - \frac{9}{2} + 9\right) = \frac{125}{6}$$

Example: If $f(x)$ is a continuous, show that: $\int_0^1 f(x) dx = \int_0^1 f(1-t) dt$

Sol:

$$\text{let } x = 1 - t \rightarrow dx = -dt$$

$$\text{at } x = 0 \rightarrow t = 1, x = 1 \rightarrow t = 0$$

$$\int_0^1 f(x) dx = \int_1^0 f(1-t) \cdot -dt = \int_0^1 f(1-t) dt$$

Method of Integration:

a) Integration formula:

$$1) \int u^n du = \frac{u^{n+1}}{n+1} + c, n \neq -1$$

$$2) \int \frac{du}{u} = \ln u + c$$

$$3) \int e^u du = e^u + c$$

$$4) \int a^u du = \frac{a^u}{\ln a} + c$$

- 5) $\int \sin u \, du = -\cos u + c$
 6) $\int \cos u \, du = \sin u + c$
 7) $\int \sec^2 u \, du = \tan u + c$
 8) $\int \csc^2 u \, du = -\cot u + c$
 9) $\int \sinh u \, du = \cosh u + c$
 10) $\int \cosh u \, du = \sinh u + c$
 11) $\int \operatorname{sech}^2 u \, du = \tanh u + c$
 12) $\int \operatorname{csch}^2 u \, du = -\coth u + c$
 13) $\int \sec u \tan u \, du = \sec u + c$
 14) $\int \csc u \cot u \, du = -\csc u + c$

Examples:

$$1) \int_0^{2\pi} \sin x \, dx = -\cos x|_0^{2\pi} = -(\cos 2\pi - \cos 0) = -(1 - 1) = 0$$

$$2) \int \sec 2x \tan 2x \, dx = \frac{1}{2} \sec(2x) + c$$

$$3) \int 5^{x^3} 15x^2 \, dx = \frac{5^{x^3}}{\ln 5} + c$$

$$4) 4 \int \operatorname{csch}^2(4x) \, dx = -\coth 4x + c$$

b) Integration substitution:

If $u = g(x)$ is a differentiable function whose range is an interval I and f is continuous on I, then

$$I = \int f[g(x)] \dot{g}(x) \, dx$$

➤ The steps to evaluate the integral is:

- 1- Substitute $u = g(x)$ and $du = g'(x) dx$ to obtain the integral $\int f(u) du$
- 2- Integrate with respect to u .
- 3- Replace u by $g(x)$ in the result.

Example: Using Substitution to find $\int \cos(7\theta + 5) d\theta$

Sol:

$$\text{let } u = 7\theta + 5 \rightarrow du = 7 d\theta \quad \therefore d\theta = \frac{1}{7} du$$

$$\begin{aligned} \int \cos(7\theta + 5) d\theta &= \int \cos u * \frac{1}{7} du = \frac{1}{7} \int \cos u du \\ &= \frac{1}{7} \sin u + c = \frac{1}{7} \sin(7\theta + 5) + c \end{aligned}$$

Example: Using Substitution to find $\int x^2 \sin(x^3) dx$

Sol:

$$\text{let } u = x^3 \rightarrow du = 3x^2 dx \quad \therefore dx = \frac{1}{3x^2} du$$

$$\begin{aligned} \int x^2 \sin(x^3) dx &= \int \sin(u) * \frac{1}{3} du = \frac{1}{3} \int \sin(u) du \\ &= -\frac{1}{3} \cos(u) + c = -\frac{1}{3} \cos(x^3) + c \end{aligned}$$

Example: Using Substitution to find $\int \frac{2z dz}{\sqrt[3]{z^2+1}}$

Sol:

$$\text{let } u = z^2 + 1 \rightarrow du = 2z dz$$

$$\frac{\int 2z dz}{\sqrt[3]{z^2+1}} = \int \frac{du}{\sqrt[3]{u}} = \int u^{-\frac{1}{3}} du = \frac{u^{\frac{2}{3}}}{\frac{2}{3}} + c = \frac{3}{2} u^{2/3} + c = \frac{3}{2} (z^2 + 1)^{\frac{2}{3}} + c$$

c) Certain power of trigonometric and hyperbolic integrals:

$$1- \int \sin^m u \cos^n u du \quad \text{or} \quad \int \sinh^m u \cosh^n u du$$

$$2- \int \tan^m u \sec^n u du \quad \text{or} \quad \int \tanh^m u \operatorname{sech}^n u du$$

$$3- \int \cot^m u \csc^n u du \quad \text{or} \quad \int \coth^m u \operatorname{csch}^n u du$$

Under type (1) there are three cases:

Case (1): If m is odd we use the identities

$$\sin^2 u = 1 - \cos^2 u \quad \text{or} \quad \sinh^2 u = \cosh^2 u - 1$$

$$\sin^3 x = \sin x \sin^2 x, \sin^5 x = \sin x \sin^4 x \text{ and so on.}$$

Example: Find $\int \sin^5 2x \cos^2 2x dx$

Sol:

$$\int \sin^5 2x \cos^2 2x dx = \int \sin^4 2x \sin 2x \cos^2 2x dx$$

$$\begin{aligned}
&= \int (\sin^2 2x)^2 \sin 2x \cos^2 2x = \int (1 - \cos^2 2x)^2 \sin 2x \cos^2 2x \\
&= \int (1 - 2 \cos^2 2x + \cos^4 2x) \sin 2x \cos^2 2x \, dx \\
&= \int \cos^2 2x (\sin 2x \, dx) - 2 \int \cos^4 2x (\sin 2x \, dx) \\
&\quad + \int \cos^6 2x (\sin 2x \, dx) \\
&= -\frac{1}{2} \frac{\cos^3 2x}{3} + \frac{2}{2} \frac{\cos^5 2x}{5} - \frac{1}{2} \frac{\cos^7 2x}{7} + c \\
&= -\frac{1}{6} \cos^3 2x + \frac{1}{5} \cos^5 2x - \frac{1}{14} \cos^7 2x + c
\end{aligned}$$

Case (2): If n is odd we use the identities

$$\cos^2 u = 1 - \sin^2 u \quad or \quad \cosh^2 u = 1 + \sinh^2 u$$

$$\cos^3 u = \cos u \cos^2 u, \quad \cosh^5 u = \cosh^4 u \cosh u \quad and \text{ so on.}$$

Example: Find $\int \sinh^4 3x \cosh^3 3x \, dx$

Sol:

$$\begin{aligned}
\int \sinh^4 3x \cosh^3 3x \, dx &= \int \sinh^4 3x \cosh^2 3x \cosh 3x \, dx \\
&= \int \sinh^4 3x (1 + \sinh^2 3x) \cosh 3x \, dx = \\
&\quad \int (\sinh^4 3x \cosh 3x + \sinh^6 3x \cosh 3x) \, dx \\
&= \frac{1}{3} \frac{\sinh^5 3x}{5} + \frac{1}{3} \frac{\sinh^6 3x}{6} + c
\end{aligned}$$

Case (3): Both n and m are even, we use the identities

$$\sin^2 u = \frac{1}{2} (1 - \cos 2u) \quad \text{or} \quad \cos^2 u = \frac{1}{2} (1 + \cos 2u)$$

Example: Find the integral $\int \sin^2 2x \cos^2 2x dx$

Sol:

$$\begin{aligned}\int \sin^2 2x \cos^2 2x dx &= \int \left[\frac{1}{2}(1 - \cos 4x) * \frac{1}{2}(1 + \cos 4x) \right] dx \\ &= \frac{1}{4} \int (1 - \cos^2 4x) dx = \frac{1}{4} [\int dx - \int \cos^2 4x] dx \\ &= \frac{1}{4} \left[x - \frac{1}{2} \int (1 + \cos 8x) dx \right] = \frac{1}{4} [x - \frac{1}{2} (\int dx + \int \cos 8x dx)] \\ &= \frac{1}{4} \left[x - \frac{1}{2} \left(x + \frac{1}{8} \sin 8x \right) \right] + c = \frac{1}{4} \left[x - \frac{x}{2} - \frac{1}{16} \sin 8x \right] + c\end{aligned}$$

➤ **Under type (2) there are two cases:**

Case (1): If n is even, we use the identities:

$$\sec^2 u = 1 + \tan^2 u \quad \text{or} \quad \operatorname{sech}^2 u = 1 - \tanh^2 u$$

Example: Find the integral $\int \operatorname{sech}^4 x \tanh x dx$

Sol:

$$\begin{aligned}\int \operatorname{sech}^4 x \tanh x dx &= \int (\operatorname{sech}^2 x \operatorname{sech}^2 x \tanh x) dx \\ &= \int [(1 - \tanh^2 x) \operatorname{sech}^2 x \tanh x] dx \\ &= \int \operatorname{sech}^2 x \tanh x dx - \int \operatorname{sech}^2 x \tanh^3 x dx \\ \\ &= \int \operatorname{sech} x (\tanh x \operatorname{sech} x dx) - \int \operatorname{sech}^2 x \tanh^2 x \tanh x dx \\ &= \int \operatorname{sech} x (\tanh x \operatorname{sech} x dx) - [\int \operatorname{sech} x (1 + \operatorname{sech}^2 x) \operatorname{sech} x \tanh x dx]\end{aligned}$$

$$\begin{aligned}
&= \int (\operatorname{sech} x \tanh x) \operatorname{sech} x dx - [\int \operatorname{sech} x (\operatorname{sech} x \tanh x dx) + \\
&\quad \int \operatorname{sech}^3 x (\operatorname{sech} x \tanh x dx)] = - \int \operatorname{sech}^3 x (\operatorname{sech} x \tanh x dx) = \frac{-\operatorname{sech}^4 x}{4} + c
\end{aligned}$$

Case (2): If n is odd, we use the identities:

$$\tan^2 u = \sec^2 u - 1 \quad \text{or} \quad \tanh^2 u = 1 - \operatorname{sech}^2 u$$

Example: Find the integral $\int \tan^3 2x \sec^2 2x dx$

Sol:

$$\begin{aligned}
\int \tan^3 2x \sec^2 2x dx &= \int \tan^2 2x \sec 2x (\tan 2x \sec 2x dx) \\
&= \int (\sec^2 2x + 1) \sec 2x (\tan 2x \sec 2x dx) \\
&= \int \sec^3 2x (\tan 2x \sec 2x dx) + \int \sec 2x (\tan 2x \sec 2x dx) \\
&= \frac{1}{2} \frac{\sec^4 2x}{4} + \frac{1}{2} \frac{\sec^2 2x}{2} + c = \frac{1}{8} \sec^4 2x + \frac{1}{4} \sec^2 2x + c
\end{aligned}$$

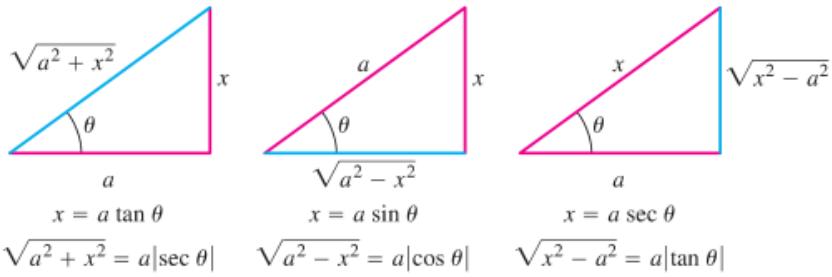
d) Trigonometric Substitutions:

Trigonometric substitutions occur when we replace the variable of integration by a trigonometric function. If the integral one of the forms:

$$a^2 + u^2, \sqrt{a^2 - u^2}, \sqrt{a^2 + u^2} \quad \text{or} \quad \sqrt{u^2 - a^2}$$

Then the substitutions as follows:

- 1) If $\sqrt{a^2 - x^2}$, $x = a \sin \theta \rightarrow a^2 - x^2 = a^2 \cos^2 \theta$
- 2) If $\sqrt{a^2 + x^2}$, $x = a \tan \theta \rightarrow a^2 + x^2 = a^2 \sec^2 \theta$
- 3) If $\sqrt{x^2 - a^2}$, $x = a \sec \theta \rightarrow x^2 - a^2 = a^2 \tan^2 \theta$



Example: Find $\int \frac{dx}{4+x^2}$

Sol:

$$x = 2 \tan \theta \rightarrow dx = 2 \sec^2 \theta \ d\theta$$

$$\tan \theta = \frac{x}{2} \rightarrow \theta = \tan^{-1}\left(\frac{x}{2}\right)$$

$$\int \frac{dx}{4+x^2} = \int \frac{2 \sec^2 \theta \ d\theta}{4+4 \tan^2 \theta} = \int \frac{2 \sec^2 \theta \ d\theta}{4 \sec^2 \theta} = \frac{1}{2} \int d\theta = \frac{1}{2} \theta + c = \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$$

Example: Find $\int_{-\frac{1}{2}}^{\sqrt{3}/2} \sqrt{1-x^2} \ dx$

Sol:

$$x = \sin \theta \rightarrow dx = \cos \theta \ d\theta$$

$$at \quad x = \frac{-1}{2} \rightarrow \sin \theta = -\frac{1}{2} \rightarrow \theta = -\frac{\pi}{6}$$

$$x = \sqrt{\frac{3}{2}} \rightarrow \sin \theta = \sqrt{\frac{3}{2}} \rightarrow \theta = -\frac{\pi}{3}$$

$$\int_{-\frac{1}{2}}^{\sqrt{3}/2} \sqrt{1-x^2} \ dx = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \sqrt{1-\sin^2 \theta} \cos \theta \ d\theta = \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \cos^2 \theta \ d\theta$$

$$\int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1+\cos 2\theta}{2} d\theta = \frac{1}{2} \int_{-\frac{\pi}{6}}^{\frac{\pi}{3}} (1 + \cos 2\theta) d\theta = \frac{1}{2} \left[\theta + \frac{1}{2} \sin 2\theta \right]_{-\frac{\pi}{6}}^{\frac{\pi}{3}} =$$

$$\frac{1}{2} \left[\frac{\pi}{3} + \frac{1}{2} \frac{\sqrt{3}}{2} + \frac{\pi}{6} + \frac{1}{2} \frac{\sqrt{3}}{2} \right] = \frac{1}{2} \left[\frac{\pi}{2} + \frac{\sqrt{3}}{2} \right] = \frac{1}{2} \frac{\pi + 3}{2} = \frac{\pi + 3}{4}$$

Example: Find $\int \frac{\sqrt{x^2 - 7}}{7} dx$

Sol:

$$x = \sqrt{7} \sec \theta \rightarrow \sec \theta = \frac{x}{\sqrt{7}}, \quad \theta = \sec^{-1} \frac{x}{\sqrt{7}}$$

$$dx = \sqrt{7} \sec \theta \tan \theta d\theta$$

$$\begin{aligned} \int \frac{\sqrt{x^2 - 7}}{7} dx &= \int \frac{\sqrt{7 \sec^2 \theta - 7}}{\sqrt{7} \sec \theta} \sqrt{7} \sec \theta \tan \theta d\theta = \\ \int \sqrt{7(\sec^2 \theta - 1)} \tan \theta d\theta &= \int \sqrt{7} \tan^2 \theta d\theta \\ &= \sqrt{7} \int (\sec^2 \theta - 1) d\theta = \sqrt{7} [\tan \theta - \theta] + c \\ &= \sqrt{7} [\tan \left(\sec^{-1} \frac{x}{\sqrt{7}} \right) - \sec^{-1} \frac{x}{\sqrt{7}}] + c = \sqrt{7} \left[\frac{1}{\sqrt{7}} \sqrt{x^2 - 7} - \sec^{-1} \frac{x}{\sqrt{7}} \right] + c \end{aligned}$$

e) Integration by part:

$$\int \mathbf{u} d\mathbf{v} = \mathbf{u} \cdot \mathbf{v} - \int \mathbf{v} d\mathbf{u}$$

Where u and v are two different functions with respect to x.

Example: Find $\int \ln x \, dx$

Sol:

$$u = \ln x \rightarrow du = \frac{1}{x} \, dx$$

$$dv = dx \rightarrow v = x$$

$$\int \ln x \, dx = x \ln x - \int x \cdot \frac{1}{x} \, dx = x \ln x - \int dx = x \ln x - x + c$$

Example: Find $\int \tan^{-1} x \, dx$

Sol:

$$u = \tan^{-1} x \rightarrow du = \frac{dx}{x^2+1}$$

$$dv = dx \rightarrow v = x$$

$$\int \ln \tan^{-1} x \, dx = x \tan^{-1} x - \int \frac{x}{x^2+1} \, dx = x \tan^{-1} x - \frac{1}{2} \ln(x^2 + 1) + c$$

Example: Find $\int x \ln x \, dx$

Sol:

$$u = \ln x \rightarrow du = \frac{1}{x} \, dx$$

$$dv = x \, dx \rightarrow v = \frac{1}{2}x^2$$

$$\int x \ln x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x \, dx$$

$$= \frac{1}{2}x^2 \ln x - \frac{1}{2} \int x \, dx = \frac{1}{2}x^2 \ln x - \frac{1}{4}x^2 + c$$

Example: Find $\int x e^x dx$

Sol:

$$u = x \rightarrow du = dx$$

$$dv = e^x \rightarrow v = e^x$$

$$\int x e^x dx = x e^x - \int e^x dx = x e^x - e^x + c$$

Example: Find $\int x^2 e^x dx$

Sol:

$$u = x^2 \rightarrow du = 2x dx$$

$$dv = e^x \rightarrow v = e^x$$

$$\int x^2 e^x dx = x^2 e^x - \int 2x e^x dx = I_1$$

$$I_1 = \int 2x e^x dx$$

$$u = 2x \rightarrow du = 2 dx$$

$$dv = e^x \rightarrow v = e^x$$

$$I_1 = 2x e^x - 2 \int e^x dx = 2x e^x - 2 e^x + c$$

$$I = \int x^2 e^x dx = x^2 e^x - (2x e^x - 2 e^x + c) = x^2 e^x - 2x e^x + 2e^x + c$$

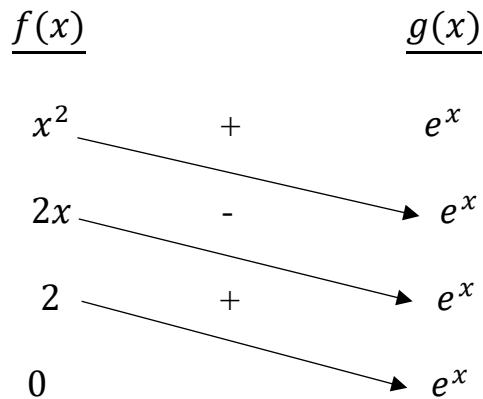
f) Tabular Integration:

We have seen that integrals of the form $\int g(x) f(x) dx$ in which f can be differentiated repeatedly to become zero and g can be integrated repeatedly without difficulty, are natural candidates for integration by parts.

Example: Find the integral $\int x^2 e^x dx$

Sol:

$$\text{let } x^2 = f(x), \quad g(x) = e^x$$



$$I = \int x^2 e^x dx = x^2 e^x - 2x e^x + 2e^x + c$$

Example: Find the integral $\int (x^3 - 2x^2 + 3x + 1) \sin 2x dx$

Sol:

$$f(x) = x^3 - 2x^2 + 3x + 1, \quad g(x) = \sin 2x$$

<u>$f(x)$</u>	<u>$g(x)$</u>
$x^3 - 2x^2 + 3x + 1$	$\sin 2x$
$3x^2 - 4x + 3$	$-\frac{1}{2}\cos 2x$
$6x - 4$	$-\frac{1}{4}\sin 2x$
6	$\frac{1}{8}\cos 2x$
0	$\frac{1}{16}\sin 2x$

$$\int (x^3 - 2x^2 + 3x + 1) \sin 2x \, dx = (x^3 - 2x^2 + 3x + 1) \left(-\frac{1}{2} \cos 2x \right) + \frac{1}{4} (3x^2 - 4x + 3)(\sin 2x) + \frac{1}{8} (6x - 4)(\cos 2x) - \frac{6}{16} \sin 2x + C$$

g) Integration of Rotational Functions:

Success in writing a rational function $\frac{f(x)}{g(x)}$ as a sum of partial fractions

depends on two things:

- The degree of $f(x)$ must be less than the degree of $g(x)$. That is, the fraction must be proper. If it isn't, divide $f(x)$ by $g(x)$ and work with the remainder term.
- We must know the factors of $g(x)$. In theory, any polynomial with real coefficients can be written as a product of real linear factors and real quadratic factors. In practice, the factors may be hard to find.

Method of Partial Fractions [$f(x) / g(x)$]:

1- Let $x - r$ be a linear factor of $g(x)$. Suppose that $(x - r)^m$ is the highest power of $x - r$ that divides $g(x)$. Then, to this factor, assign the sum of the m partial fractions:

$$\frac{A_1}{x - r} + \frac{A_2}{(x - r)^2} + \cdots + \frac{A_m}{(x - r)^m}$$

2- Let $x^2 + px + q$ be a quadratic factor of $g(x)$. Suppose that $(x^2 + px + q)^m$ is the highest power of this factor that divides $g(x)$. Then, to this factor, assign the sum of the n partial fractions

$$\frac{B_1 x + C_1}{x^2 + px + q} + \frac{B_2 x + C_2}{(x^2 + px + q)^2} + \cdots + \frac{B_n x + C_n}{(x^2 + px + q)^m}$$

3- Set the original fraction $f(x) / g(x)$ equal to the sum of all these partial fractions. Clear the resulting equation of fractions and arrange the terms in decreasing powers of x .

4- Equate the coefficients of corresponding powers of x and solve the resulting equations for the undetermined coefficients.

Where A, B, C are constants and must be found.

Example: Find $\int \frac{x^2+4x+1}{(x-1)(x+1)(x+3)} dx$

Sol:

$$\frac{x^2+4x+1}{(x-1)(x+1)(x+3)} = \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} = \frac{A(x+1)(x+3)+B(x-1)(x+3)+C(x-1)(x+1)}{(x-1)(x+1)(x+3)}$$

$$\frac{A(x^2+3x+x+3)+B(x^2+3x-x-3)+C(x^2+x-x-1)}{(x-1)(x+1)(x+3)} = \frac{A(x^2+4x+3)+B(x^2+2x-3)+C(x^2-1)}{(x-1)(x+1)(x+3)}$$

$$\frac{Ax^2+4Ax+3A+Bx^2+2Bx-3B+Cx^2-C}{(x-1)(x+1)(x+3)} = \frac{x^2(A+B+C)+x(4A+2B)+3A-3B-C}{(x-1)(x+1)(x+3)}$$

$$x^2 + 4x + 1 = x^2(A + B + C) + x(4A + 2B) + 3A - 3B - C$$

$$x^2 = x^2(A + B + C) \rightarrow A + B + C = 1 \dots (1)$$

$$4x = x(4A + 2B) \rightarrow 4A + 2B = 4 \dots (2)$$

$$1 = 3A - 3B - C \rightarrow 3A - 3B - C = 1 \dots (3)$$

$$A + B + C = 1 \dots (1)$$

$$\underline{3A - 3B - C = 1 \dots (3)}$$

$$\underline{4A - 2B = 2} \dots (4)$$

We can solve equation (2) with (4) to obtain the value of A

$$4A + 2B = 4 \dots (2)$$

$$\underline{4A - 2B = 2} \dots (4)$$

$$8A = 6 \rightarrow A = \frac{6}{8} = \frac{3}{4}$$

substitute the value of A in equation (2) we obtian

$$4 * \frac{3}{4} + 2B = 4 \rightarrow 3 + 2B = 4 \rightarrow 2B = 4 - 3 = 1 \rightarrow B = \frac{1}{2}$$

Put the values of A and B in equation (1) or (3) to find C

$$A + B + C = 1 \dots (1) \rightarrow \frac{3}{4} + \frac{1}{2} + C = 1 \rightarrow \frac{3+2}{4} + C = 1$$

$$\frac{5}{4} + C = 1 \rightarrow C = 1 - \frac{5}{4} = -\frac{1}{4}$$

Then;

$$\int \frac{A}{(x-1)} + \frac{B}{(x+1)} + \frac{C}{(x+3)} dx = \int \frac{3/4}{(x-1)} + \frac{1/2}{(x+1)} - \frac{1/4}{(x+3)} dx$$

$$\frac{3}{4} \int \frac{dx}{x-1} + \frac{1}{2} \int \frac{dx}{x+1} - \frac{1}{4} \int \frac{dx}{x+3} = \frac{3}{4} \ln(x-1) + \frac{1}{2} \ln(x+1) - \frac{1}{4} \ln(x+3) + c$$

Example: Evaluate $\int \frac{6x+7}{(x+2)^2} dx$

Sol:

$$\frac{6x+7}{(x+2)^2} = \frac{A}{x+2} + \frac{B}{(x+2)^2} = \frac{A(x+2)+B}{(x+2)^2} = \frac{Ax+2A+B}{(x+2)^2}$$

$$6x+7 = Ax+2A+B$$

$$6x = Ax \rightarrow 6 = A \quad \dots (1)$$

$$7 = 2A + B \quad \dots (2) \text{ put the value of } A \text{ in equation (2)}$$

$$7 = 2 * 6 + B = 12 + B \rightarrow B = 7 - 12 = -5$$

$$\int \frac{6x+7}{(x+2)^2} dx = \int \frac{A}{x+2} dx + \int \frac{B}{(x+2)^2} dx = \int \frac{6}{x+2} dx + \int \frac{-5}{(x+2)^2} dx$$

$$6 \int \frac{dx}{x+2} - 5 \int \frac{dx}{(x+2)^2} = 6 \ln(x+2) - 5(x+2)^{-1} + c$$

Example: Evaluate $\int \frac{dx}{x(x^2+1)^2}$

Sol:

$$\frac{dx}{x(x^2+1)^2} = \frac{A}{x} + \frac{Bx+C}{(x^2+1)} + \frac{Dx+E}{(x^2+1)^2}$$

Multiplying by $x(x^2 + 1)^2$ and equate numerators we obtain:

$$1 = A(x^2 + 1)^2 + (Bx + C)x(x^2 + 1) + (Dx + E)(x)$$

$$1 = A(x^4 + 2x^2 + 1) + (Bx + C)(x^3 + x) + Dx^2 + Ex$$

$$1 = A(x^4 + 2x^2 + 1) + (Bx + C)(x^3 + x) + Dx^2 + Ex$$

$$1 = Ax^4 + 2Ax^2 + A + Bx^4 + Bx^2 + Cx^3 + Cx + Dx^2 + Ex$$

$$1 = x^4(A + B) + x^3(C) + x^2(2A + B + D) + x(C + E) + A$$

$$0 = x^4(A + B) \rightarrow A + B = 0 \rightarrow A = -B \dots (1)$$

$$0 = x^3(C) \rightarrow C = 0 \dots (2)$$

$$0 = x^2(2A + B + D) \rightarrow 2A + B + D = 0 \dots (3)$$

$$0 = x(C + E) \rightarrow C + E = 0 \rightarrow C = -E \dots (4)$$

$$1 = A \rightarrow \text{from (1)} \quad A = -B \quad \text{then } B = -1$$

And from (3) we find the value of D

$$2A + B + D = 0 \rightarrow 2 * 1 + (-1) + D = 0 \rightarrow 2 - 1 + D = 0$$

$$1 + D = 0 \rightarrow D = -1$$

And from (4) we find the value of E

$$C = -E \dots (4) \rightarrow C = 0 \quad \text{then } E = 0$$

$$\int \frac{dx}{x(x^2+1)^2} = \int \frac{A}{x} dx + \int \frac{Bx+C}{(x^2+1)} dx + \int \frac{Dx+E}{(x^2+1)^2} dx$$

$$\int \frac{1}{x} dx + \int \frac{-x}{(x^2+1)} dx + \int \frac{-x}{(x^2+1)^2} dx = \ln x - \frac{1}{2} \ln(x^2+1) - \int \frac{u du}{u^2} = \ln x - \frac{1}{2} \ln(x^2+1) - \frac{1}{2} \ln u^2 = \ln x - \frac{1}{2} \ln(x^2+1) - \frac{1}{2} \ln(x^2+1)^2 + c$$

Example: Find the value of A, B, and C in $\int \frac{x^2+1}{(x-1)(x-2)(x-3)} dx$

(Assigning numerical values to x)

Sol:

$$\frac{x^2+1}{(x-1)(x-2)(x-3)} = \frac{A}{x-1} + \frac{B}{x-2} + \frac{C}{x-3} = \frac{A(x-2)(x-3)+B(x-1)(x-3)+C(x-1)(x-2)}{(x-1)(x-2)(x-3)}$$

$$x^2 + 1 = A(x-2)(x-3) + B(x-1)(x-3) + C(x-1)(x-2)$$

Then let $x = 1, 2, 3$ successively to find A, B, and C

$$x = 1 \rightarrow (1)^2 + 1 = A(1-2)(1-3) + B(0) + C(0) = A(-1) * (-2)$$

$$2 = 2A \rightarrow A = \frac{2}{2} = 1$$

$$x = 2 \rightarrow (2)^2 + 1 = A(0) + B(2-1)(2-3) + C(0) = B(1) * (-1)$$

$$5 = -B \rightarrow B = -5$$

$$x = 3 \rightarrow (3)^2 + 1 = A(0) + B(0) + C(3-1)(3-2) = C(2)(1)$$

$$10 = 2C \rightarrow C = \frac{10}{2} = 5$$

$$\begin{aligned} \int \frac{A}{(x-1)} dx + \int \frac{B}{(x-2)} dx + \int \frac{C}{(x-3)} dx &= \int \frac{1}{(x-1)} dx + \int \frac{-5}{(x-2)} dx + \int \frac{5}{(x-3)} dx \\ &= \int \frac{dx}{(x-1)} - 5 \int \frac{dx}{(x-2)} + 5 \int \frac{dx}{(x-3)} = \ln(x-1) - 5 \ln(x-2) + 5 \ln(x-3) + \end{aligned}$$

HOMEWORK

Find the following integrals

$$1- \int \frac{2y}{y^2 - 25} dy$$

$$2- \int \sin^4 2x dx$$

$$3- \int (e^{3x} + 5e^{-x}) dx$$

$$4- \int x^2 e^x dx$$

$$5- \int \frac{x}{x^2 + 4x + 3} dx$$

$$7- \int \frac{dx}{x(x^2 + 1)^2}$$

$$8- \int \frac{(2x^2 - 3x + 2)dx}{(x-1)^2(x-2)}$$