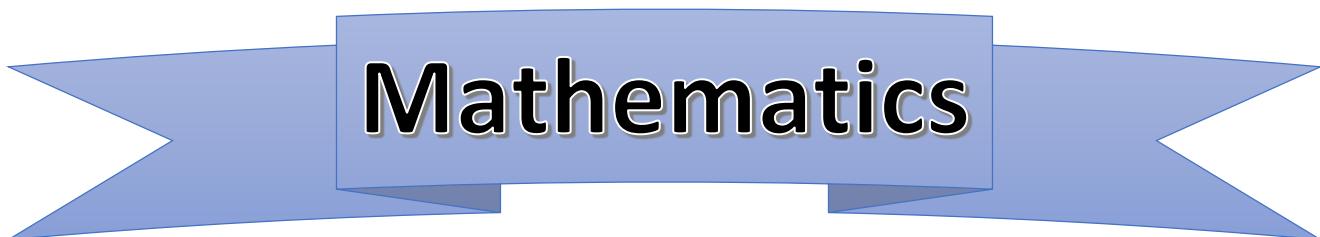


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# Chapter One

## Vectors

**Vectors:** is a quantity which has magnitude and direction such as force, displacement and velocity quantities.

### Vectors in The Plane and The Space:

- The vectors  $\vec{v}$  in the plane takes the form:  $\vec{v} = a_1\mathbf{i} + a_2\mathbf{j}$
- The vectors  $\vec{v}$  in the space takes the form:  $\vec{v} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$

Where  $a_1$ ,  $a_2$ , and  $a_3$  are called the vector components in the direction of  $\mathbf{i}$ ,  $\mathbf{j}$ ,  $\mathbf{k}$ .

### Operation on Vectors:

#### 1- Addition:

- For any vectors :  $\vec{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\vec{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$

Then,

$$\vec{A} + \vec{B} = (a_1 + b_1)\mathbf{i} + (a_2 + b_2)\mathbf{j} + (a_3 + b_3)\mathbf{k}$$

**Example:** If  $\vec{A} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $\vec{B} = \mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$  then

$$\vec{A} + \vec{B} = (2 + 1)\mathbf{i} + (1 + 9)\mathbf{j} + (3 + 4)\mathbf{k} = 3\mathbf{i} + 10\mathbf{j} + 7\mathbf{k}$$

#### 2- Subtraction:

$$\vec{A} - \vec{B} = (a_1 - b_1)\mathbf{i} + (a_2 - b_2)\mathbf{j} + (a_3 - b_3)\mathbf{k}$$

**Example:** If  $\vec{A} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $\vec{B} = \mathbf{i} + 9\mathbf{j} + 4\mathbf{k}$  then

$$\vec{A} - \vec{B} = (2 - 1)\mathbf{i} + (1 - 9)\mathbf{j} + (3 - 4)\mathbf{k} = \mathbf{i} - 8\mathbf{j} - \mathbf{k}$$

### 3- Multiplication by scalars:

For any constant  $c$ , then

$$c\vec{A} = (ca_1)\mathbf{i} + (ca_2)\mathbf{j} + (ca_3)\mathbf{k}$$

**Example:** If  $\vec{A} = 2\mathbf{i} + \mathbf{j} + 3\mathbf{k}$  and  $c=4$  then

$$c\vec{A} = 4(2\mathbf{i} + \mathbf{j} + 3\mathbf{k}) = 8\mathbf{i} + 4\mathbf{j} + 12\mathbf{k}$$

### 4- Dot product of two vectors:

If  $\vec{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  and  $\vec{B} = b_1\mathbf{i} + b_2\mathbf{j} + b_3\mathbf{k}$  then;

$$\vec{A} \cdot \vec{B} = a_1b_1 + a_2b_2 + a_3b_3$$

**Example:** If  $\vec{A} = 2\mathbf{i} + 4\mathbf{j} - \mathbf{k}$  and  $\vec{B} = \mathbf{i} - 2\mathbf{j} + 7\mathbf{k}$ , find  $\vec{A} \cdot \vec{B}$

$$\text{Sol: } \vec{A} \cdot \vec{B} = (2 * 1) + (4 * -2) + (-1 * 7) = 2 - 8 - 7 = -13$$

**Theorem:** Let  $\theta$  be the angle between the vectors  $A$  and  $B$  then;

$$\vec{A} \cdot \vec{B} = |A||B| \cos \theta$$

The angle between the

$$\theta = \cos^{-1}\left(\frac{\vec{A} \cdot \vec{B}}{|A||B|}\right)$$

### 5- Cross product:

For any vectors A and B we define cross product of them as follows:

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ a_1 & a_2 & a_3 \\ b_1 & b_2 & b_3 \end{vmatrix} = i(a_2 a_3 - b_2 b_3) - j(a_1 a_3 - b_1 b_3) + k(a_1 a_2 - b_1 b_2)$$

**Example:** If  $\vec{A} = 2i + 4j - k$  and  $\vec{B} = i - 2j + 7k$ , find  $\vec{A} \times \vec{B}$

**Sol:**

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ 2 & 4 & -1 \\ 1 & -2 & 7 \end{vmatrix} = i(4 - (-1)) - j(2 - 7) + k(2 - 4) = 5i + 5j - 2k$$

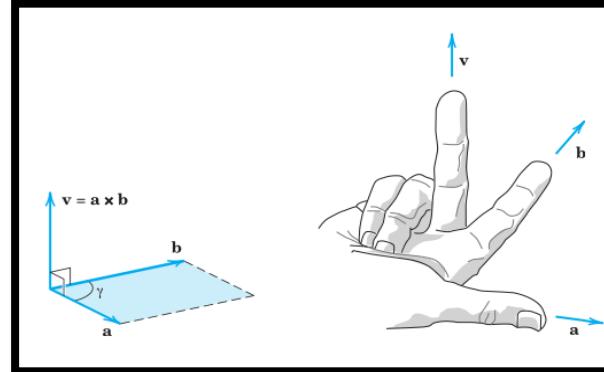
$$\vec{A} \times \vec{B} = i(28 - 2) - j(14 + 1) + k(-4 - 4) = 26i - 15j + 0k$$

**Theorem:** Let  $\theta$  be the angle between the vectors A and B then;

$$\vec{A} \times \vec{B} = |A||B| \sin \theta \ \hat{n}$$

$$|\vec{A} \times \vec{B}| = |A||B| \sin \theta$$

$$\sin \theta = \frac{|\vec{A} \times \vec{B}|}{|A||B|}$$



### The vectors between two points:

The vector between the first point  $p_1(x_1, y_1, z_1)$  to second point  $p_2(x_2, y_2, z_2)$  is:

$$\overrightarrow{p_1 p_2} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

**Example:** Find the length of the vector  $\vec{a}$  with initial point p (3, 1, 4) and terminal point q(1, -2, 4).

Sol:

$$\overrightarrow{pq} = (x_2 - x_1)i + (y_2 - y_1)j + (z_2 - z_1)k$$

$$\overrightarrow{pq} = \vec{a} = (1 - 3)i + (-2 - 1)j + (4 - 4)k$$

$$\vec{a} = -2i - 3j + 0k$$

**Length and Direction:**

If  $\vec{A} = a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}$  then the length of A is:

$$|A| = |a_1\mathbf{i} + a_2\mathbf{j} + a_3\mathbf{k}| = \sqrt{a_1^2 + a_2^2 + a_3^2}$$

But the direction of A is given by

$$\mathbf{v} = \frac{\vec{v}}{|v|} \quad \text{or} \quad A = \frac{\vec{A}}{|A|}$$

**Example:** Find the direction vector of  $\vec{v} = 3\mathbf{i} + 4\mathbf{j}$ .

Sol:

$$u = \frac{\vec{v}}{|v|} \quad (\text{unit vector or direction})$$

$$|v| = \sqrt{9 + 16} = \sqrt{25} = 5$$

$$\text{Then; } u = \frac{3\mathbf{i} + 4\mathbf{j}}{5} = \frac{3}{5}\mathbf{i} + \frac{4}{5}\mathbf{j}$$

$$\theta = \tan^{-1} \frac{y}{x} = \frac{4}{3} = 53.13$$

**Example:** If  $\vec{A} = 3\mathbf{i} - 4\mathbf{k}$  and  $\vec{B} = 7\mathbf{j} + 2\mathbf{k}$ , find  $\vec{A} + \vec{B}$  and  $3\vec{A} - 4\vec{B}$ .

Sol:

$$\vec{A} + \vec{B} = (3i - 4k) + (7j + 2k) = 3i + 7j + (-4 + 2)k = 3i + 7j - 2k$$

$$3\vec{A} = 3(3i - 4k) = 9i - 12k$$

$$4\vec{B} = 4(7j + 2k) = 28j + 8k$$

$$3\vec{A} - 4\vec{B} = (9i - 12k) - (28j + 8k) = 9i - 28j - 20k$$

**Example:** If  $\vec{A} = 3i + 2j - 4k$  and  $\vec{B} = i + 2j - k$  then find

$$(1) \vec{A} \cdot \vec{B}$$

$$(2) \vec{A} \cdot \vec{B} \text{ if } \theta = 30$$

Sol:

$$(1) \vec{A} \cdot \vec{B} = 3(1) + 2(2) - 4(-1) = 3 + 4 + 4 = 11$$

$$(2) \vec{A} \cdot \vec{B} = |A||B| \cos \theta$$

$$\vec{A} \cdot \vec{B} = \sqrt{9+4+16} \sqrt{1+4+1} \cos 30$$

$$\vec{A} \cdot \vec{B} = \sqrt{29} \sqrt{6} (0.866) = 5.3 * 2.4 * 0.866 = 11.015$$

<b>Note:</b> $i \cdot i = j \cdot j = k \cdot k = 1$
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**Proof:**

$$i \cdot i = |i||i| \cos 0 = 1 \cdot 1 \cos 0 = 1$$

This can be applied on  $j \cdot j$  and  $k \cdot k$

<b>But:</b>
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$i \cdot j = i \cdot k = j \cdot k = 0$
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**Proof:**

$$\mathbf{i} \cdot \mathbf{j} = |\mathbf{i}| |\mathbf{j}| \cos \theta = 1.1 \cos 90 = 0$$

$$\mathbf{i} \cdot \mathbf{k} = |\mathbf{i}| |\mathbf{k}| \cos \theta = 1.1 \cos 90 = 0$$

The vector  $\mathbf{i}$  is normal to the vectors  $\mathbf{j}$  and  $\mathbf{k}$  and the vector  $\mathbf{j}$  is normal to the vectors  $\mathbf{k}$  and  $\mathbf{i}$ , and the vector  $\mathbf{k}$  is normal to the vectors  $\mathbf{i}$  and  $\mathbf{j}$ .

**Example:**

Find the angle between the vectors  $\vec{A} = 3\mathbf{i} + 5\mathbf{k}$  and  $\vec{B} = 2\mathbf{i} - 3\mathbf{j}$

Sol:

$$\cos \theta = \frac{\vec{A} \cdot \vec{B}}{|A||B|}$$

$$\vec{A} \cdot \vec{B} = 3(2) + 0(-3) + 5(0) = 6$$

$$|A| = \sqrt{9 + 0 + 25} = \sqrt{34}$$

$$|B| = \sqrt{4 + 0 + 9} = \sqrt{13}$$

$$\theta = \cos^{-1}\left(\frac{6}{\sqrt{34}\sqrt{13}}\right) = 71$$

**Not:** Any two vectors be perpendicular if and only if  $\vec{A} \cdot \vec{B} = 0 \rightarrow \theta = 90^\circ$

**Example:** Is  $\vec{A} = 3i - 4j$  orthogonal on  $\vec{B}$  where  $\vec{B} = 4i + 3j$  ?

Sol:

$$\vec{A} \cdot \vec{B} = 3(4) - 4(3) = 12 - 12 = 0$$

$$\therefore \vec{A} \perp \vec{B}, \theta = 90^\circ$$

**Example:** If  $\vec{A} = 3ai + 4j - k$  and  $\vec{B} = 4i + j - 3k$ , find the value of a if  $\vec{A}$  normal to  $\vec{B}$ .

Sol:

$$\vec{A} \perp \vec{B} \rightarrow \vec{A} \cdot \vec{B} = 0$$

$$\vec{A} \cdot \vec{B} = 12a + 4 + 3$$

$$0 = 12a + 7$$

$$12a = -7$$

$$\therefore a = -\frac{7}{12}$$

**Note:**

$$\vec{A} \times \vec{B} = -\vec{B} \times \vec{A}, \quad i \times i = j \times j = k \times k = 0$$

**Proof:**

$$\mathbf{i} \times \mathbf{i} = |\mathbf{i}||\mathbf{i}| \sin \theta = 1.1 \sin 0 = 0$$

This can be applied on  $\mathbf{j} \times \mathbf{j}$  and  $\mathbf{k} \times \mathbf{k}$

$$\mathbf{i} \times \mathbf{j} = \mathbf{k}, \mathbf{j} \times \mathbf{k} = \mathbf{i}, \mathbf{k} \times \mathbf{i} = \mathbf{j}$$

**Example:** If  $\vec{A} = 3\mathbf{i} - 4\mathbf{j} + \mathbf{k}$  and  $\vec{B} = \mathbf{j} + \mathbf{k}$  then find  $\vec{A} \times \vec{B}$ .

Sol:

$$\vec{A} \times \vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & -4 & 1 \\ 0 & 1 & 1 \end{vmatrix}$$

$$\vec{A} \times \vec{B} = \mathbf{i}(-5) - \mathbf{j}(3) + \mathbf{k}(3) = -5\mathbf{i} - 3\mathbf{j} + 3\mathbf{k}$$

**Example:** Find the unit vector of the intersection  $\vec{A} = 3\mathbf{i} + 4\mathbf{j} + \mathbf{k}$  with

$$\vec{B} = \mathbf{i} - 3\mathbf{j}.$$

Sol:

$$u = \frac{\vec{v}}{|v|} = \frac{\vec{A} \times \vec{B}}{|A \times B|}$$

$$\vec{A} \times \vec{B} = \begin{vmatrix} \mathbf{i} & \mathbf{j} & \mathbf{k} \\ 3 & 4 & 1 \\ 1 & -3 & 0 \end{vmatrix} = 3\mathbf{i} + \mathbf{j} - 13\mathbf{k}$$

$$|A \times B| = \sqrt{9 + 1 + 169} = \sqrt{179}$$

$$u = \frac{\vec{v}}{|v|} = \frac{\vec{A} \times \vec{B}}{|\vec{A} \times \vec{B}|} = \frac{3i + j - 13k}{\sqrt{179}}$$

**Not:** Any two vectors be parallel if and only if  $\vec{A} \times \vec{B} = 0 \rightarrow \theta = 0$

**Example:** Proof that  $\vec{A}$  parallel to  $\vec{B}$  if  $\vec{A} = -i - j - k$  and  $\vec{B} = i + j + k$ .

Sol:

$$\vec{A} \times \vec{B} = \begin{vmatrix} i & j & k \\ -1 & -1 & -1 \\ 1 & 1 & 1 \end{vmatrix} = i(-1 + 1) - j(-1 + 1) + k(-1 + 1) = 0$$

$$\therefore \vec{A} \parallel \vec{B}$$

## Homework

1- Find the vector  $\vec{v}$  between the points  $P_1(8,3,-2)$  and  $P_2(5,0,7)$  and then find the unit vector of it.

2- Find the vector that is perpendicular to both

$$\vec{A} = i + j + k \text{ and } \vec{B} = i + j .$$

3- If  $\vec{A} - 4\vec{B} = 2i + 4k$  and  $2\vec{A} - 2\vec{B} = 5i - 7k$ , find:

(a)  $\vec{A}$  and  $\vec{B}$

(b)  $\vec{A} \cdot \vec{B}$

(c) Angle between  $\vec{A}$  and  $\vec{B}$

(d)  $\vec{A} \cdot (\vec{A} \times \vec{B})$

4-Let  $\vec{A} = i + j - 4k$ ,  $\vec{B} = i - j + 4k$ ,  $\vec{C} = i + 4k$ .

Which vectors, if any, are (a) perpendicular? (b) Parallel?

# Chapter Two

## Complex Numbers

**Definition:** Set of complex numbers  $z$  defined by:

$$z = \{x, iy \mid x, y \in R, i = \sqrt{-1}\}$$

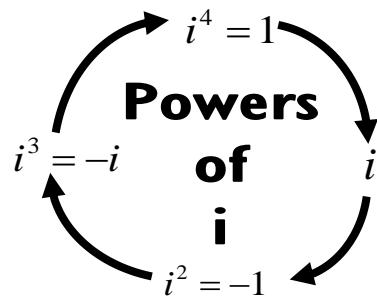
Therefore:

$$z = x + iy, \quad i^2 = -1$$

$x$  = real  $z$  ,  $y$  = imaginary of  $z$

The first four powers of  $i$  establish an important pattern and should be memorized.

### **Powers of $i$ :**



$$i^1 = i \quad i^2 = -1 \quad i^3 = -i \quad i^4 = 1$$

**Divide the exponent by 4**

No remainder: answer is 1.

remainder of 1: answer is  $i$ .

remainder of 2: answer is  $-1$ .

remainder of 3: answer is  $-i$ .

Express these numbers in terms of  $i$ .

$$1.) \sqrt{-5} = \sqrt{-1 * 5} = \sqrt{-1} \sqrt{5} = i\sqrt{5}$$

$$2.) -\sqrt{-7} = -\sqrt{-1 * 7} = -\sqrt{-1} \sqrt{7} = -i\sqrt{7}$$

$$3.) \sqrt{-99} = \sqrt{-1 * 99} = \sqrt{-1} \sqrt{99} \\ = i\sqrt{3 \cdot 3 \cdot 11} = 3i\sqrt{11}$$

**Complex Number:**

A complex number  $\mathbf{z}$  is an ordered pair of real numbers  $[a, b] = a + ib$  where  $a$  is the real part of  $\mathbf{z}$  and  $b$  is the imaginary part of  $\mathbf{z}$ .

**Complex Conjugate:**

The complex conjugate of a complex number  $z = a - ib$  (replace  $i$  with  $-i$  only).

**Example:**  $z^* = \bar{z} = \overline{a + ib} = a - ib$

## Properties of Complex Numbers:

Given two complex numbers

$$z_1 = a_1 + b_1, z_2 = a_2 + b_2 ; \text{ Then}$$

### 1- Equality:

$$z_1 = z_2 \text{ if and only if } a_1 = a_2, b_1 = b_2$$

### 2- Addition and Subtraction:

$$z_1 \pm z_2 = (a_1 + ib_1) \pm (a_2 + ib_2) = (a_1 \pm a_2) \pm i(b_1 \pm b_2)$$

### 3- Multiplication:

$$\begin{aligned} z_1 \cdot z_2 &= (a_1 + ib_1) \cdot (a_2 + ib_2) \\ &= a_1a_2 + ia_1b_2 + a_2(ib_1) + (ib_1)(ib_2) \\ &= (a_1a_2 - b_1b_2) + i(a_1b_2 + a_2b_1) \end{aligned}$$

### 4- Multiply constant by z:

Let  $k$  be a constant and  $z$  a complex number then:

$$k \cdot z = k(a_1 + ib_1) = ka_1 + ikb_1$$

### 5- Opposite complex number:

$$z_1 = a_1 + ib_1, z_2 = -a_1 - ib_1$$

$$z_1 + z_2 = (a_1 + ib_1) + (-a_1 - ib_1) = 0 + 0i = 0$$

### 6- Division:

$$\frac{z_1}{z_2} = \frac{a_1 + ib_1}{a_2 + ib_2}$$

$$\frac{z_1}{z_2} = \frac{z_1}{z_2} * \frac{z_2^*}{z_2^*} = \frac{a_1+i b_1}{a_2+i b_2} * \frac{a_2-i b_2}{a_2-i b_2} = \frac{(a_1 a_2 + b_1 b_2) + i(b_1 a_2 - a_1 b_2)}{a_2^2 + b_2^2}$$

$$\operatorname{Re} \left\{ \frac{z_1}{z_2} \right\} = \frac{(a_1 a_2 + b_1 b_2)}{a_2^2 + b_2^2}$$

$$\operatorname{Im} \left\{ \frac{z_1}{z_2} \right\} = \frac{b_1 a_2 - a_1 b_2}{a_2^2 + b_2^2}$$

### 7- Modulus of complex number:

If  $z = x + iy$  then the absolute of  $z$  is

$$|z| = \sqrt{x^2 + y^2}, \quad |\bar{z}| = |\bar{z}|$$

**Examples:** If  $z = 3 + 4i$  then:

$$1- |z| = \sqrt{3^2 + 4^2} = \sqrt{25} = 5$$

$$2- |2 - 2i| = \sqrt{2^2 + 2^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$$3 - (2 + 3i) + (4 + 5i) = 6 + 8i$$

$$4- \overline{2 + 3i} = 2 - 3i$$

5- Let  $z = 5 - 2i$ ,  $w = 3 + i$  find:

a)  $|z| \cdot |w| ; |\bar{z}| \cdot |\bar{w}|$

b)  $\frac{1}{w} ; \frac{\bar{w}}{w\bar{w}}$

c)  $\frac{z}{w} ; z \cdot \bar{z} ; \overline{z \cdot w}$

**Sol:**

a)  $|z| = \sqrt{5^2 + 2^2} = \sqrt{29}$

$$|w| = \sqrt{3^2 + 1^2} = \sqrt{10}$$

$$|\bar{w}| = |\overline{3+i}| = |3-i| = \sqrt{3^2 + (-1)^2} = \sqrt{10}$$

$$|\bar{z}| = |z| = |\overline{5-2i}| = |5-2i| = \sqrt{5^2 + (-2)^2} = \sqrt{29}$$

$$|z| \cdot |w| = \sqrt{29} \cdot \sqrt{10} = \sqrt{290}$$

$$|\bar{z}| \cdot |\bar{w}| = \sqrt{29} \cdot \sqrt{10} = \sqrt{290}$$

$$\text{b) } \frac{1}{w} = \frac{1}{3+i} * \frac{3-i}{3-i} = \frac{3-i}{3^2+1^2} = \frac{3-i}{10} = \frac{3}{10} - \frac{1}{10} i$$

$$\frac{\bar{w}}{w\bar{w}} = \frac{3-i}{(3+i)(3-i)} = \frac{3-i}{3^2+1^2} = \frac{3-i}{10} = \frac{3}{10} - \frac{1}{10} i$$

$$\text{c) } \frac{z}{w} = \frac{5-2i}{3+i} = z \cdot \frac{1}{w} = (5-2i) \left( \frac{3}{10} - \frac{1}{10} i \right)$$

$$= \frac{15}{10} - \frac{5}{10} i - \frac{6}{10} i + \frac{2}{10} (i)^2 = \frac{13}{10} - \frac{11}{10} i$$

$$z \cdot \bar{z} = (5-2i)(5+2i) = 25 + 10i - 10i + 4 = 29$$

$$\overline{z \cdot w} = \overline{(5-2i)(3+i)} = \overline{(15+5i-6i+2)} = \overline{17-i} = 17+i$$

### Another properties of complex numbers:

1)  $z \cdot \bar{z} = |z|^2$

2)  $|\bar{z}| = |z|$

3)  $\frac{1}{z} = \frac{\bar{z}}{z \cdot \bar{z}}$

4)  $\bar{\bar{z}} = z$

5)  $|z| \cdot |w| = |z \cdot w|$

6)  $|z + w| = \bar{z} + \bar{w}$

7)  $\overline{z \cdot w} = \bar{z} \cdot \bar{w}$

8)  $\frac{\bar{z}}{\bar{w}} = \left( \frac{\bar{z}}{w} \right)$

9)  $\operatorname{Re}\{z\} = \frac{1}{2}(z + \bar{z})$ ,  $\operatorname{Im}\{z\} = \frac{1}{2i}(z - \bar{z})$

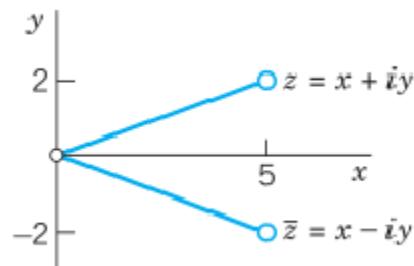
### Graphical Representation of Complex Number:

Using the Cartesian coordinates {x,y}, the x-axis is the real axis and the y- axis is the imaginary axis.

### Graphical Representation of $z^*$ and $\bar{z}$ :

In the complex plan  $z = x + iy$

And  $\bar{z} = x - iy$  is the conjugate of  $z$ .



### Polar Form of Complex Number:

We use polar coordinates  $(\theta, r)$  from  $z$  numbers.

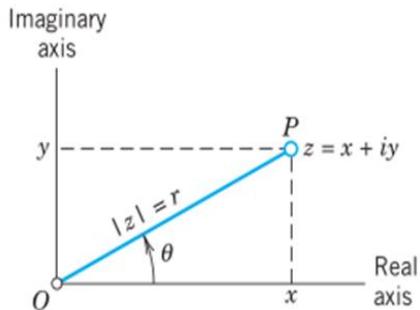
$$e^{i\theta} = \cos \theta + i \sin \theta$$

Polar coordinates.

$$\sin \theta = \frac{y}{r} \rightarrow y = r \sin \theta$$

$$\cos \theta = \frac{x}{r} \rightarrow x = r \cos \theta$$

$$z = x + iy = r \cos \theta + i r \sin \theta$$



$$z = r (\cos \theta + i \sin \theta)$$

Polar form

**Example:** Write  $-2 + 2i$  in polar form.

Sol:

$$z = -2 + 2i$$

$$|z| = r = \sqrt{(-2)^2 + 2^2} = \sqrt{8} = 2\sqrt{2}$$

$$\arg(z) = \theta = \tan^{-1} \frac{y}{x} = \tan^{-1} -\frac{2}{2} = -\frac{\pi}{4}$$

$$z = -2 + 2i = 2\sqrt{2} \left( \cos \frac{\pi}{4} - i \sin \frac{\pi}{4} \right)$$

### Multiplication and Division of Polar Form:

Let  $z_1 = r_1(\cos \theta_1 + i \sin \theta_1)$  and  $z_2 = (\cos \theta_2 + i \sin \theta_2)$

#### 1) Multiplication law:

$$z_1 z_2 = r_1(\cos \theta_1 + i \sin \theta_1) r_2 (\cos \theta_2 + i \sin \theta_2)$$

$$z_1 z_2 = r_1 r_2 [\cos(\theta_1 + \theta_2) + i \sin(\theta_1 + \theta_2)]$$

**2) Division law:**

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta_1 - \theta_2) + i \sin(\theta_1 - \theta_2)]$$

**3) Conjugate of z:**

The conjugate of z in polar form  $r(\cos \theta + i \sin \theta)$  is:

$$z^* = r[(\cos(-\theta) + i \sin(-\theta))] = r(\cos \theta - i \sin \theta)$$

### **De-Moivres' Theorem:**

$$z^n = [r(\cos \theta + i \sin \theta)]^n$$

$$z^n = r^n[\cos \theta + i \sin \theta]^n = r^n[\cos(n\theta) + i \sin(n\theta)]$$

For example take  $r = 1$

$$(\cos \theta + i \sin \theta)^n = \cos(n\theta) + i \sin(n\theta)$$

**Example:** Simplify:  $(1 + i)^n + (1 - i)^n$ .

Sol:

For  $1 + i$

$$r = \sqrt{x^2 + y^2} = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{1}{1} = \frac{\pi}{4}$$

$$z = r(\cos \theta + i \sin \theta) = \sqrt{2} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

$$\therefore 1+i = \sqrt{2} (\cos \frac{\pi}{4} + i \sin \frac{\pi}{4})$$

For  $1-i$

$$r = \sqrt{1+1} = \sqrt{2}, \theta = \tan^{-1} \frac{-1}{1} = -\frac{\pi}{4}$$

$$1-i = \sqrt{2} [\cos(-\frac{\pi}{4}) + i \sin(-\frac{\pi}{4})]$$

$$\therefore (1+i)^n = (\sqrt{2})^n \left[ \cos \frac{\pi}{4} + i \sin \frac{\pi}{4} \right]^n$$

$$(1+i)^n = 2^{\frac{n}{2}} \left[ \cos(\frac{n\pi}{4}) + i \sin(\frac{n\pi}{4}) \right]$$

$$(1-i)^n = (\sqrt{2})^n \left[ \cos \frac{-\pi}{4} + i \sin \frac{-\pi}{4} \right]^n$$

$$(1-i)^n = 2^{\frac{n}{2}} \left[ \cos(\frac{n\pi}{4}) - i \sin(\frac{n\pi}{4}) \right]$$

$$(1+i)^n + (1-i)^n =$$

$$2^{\frac{n}{2}} \left[ \cos(\frac{n\pi}{4}) + i \sin(\frac{n\pi}{4}) \right] + 2^{\frac{n}{2}} \left[ \cos(\frac{n\pi}{4}) - i \sin(\frac{n\pi}{4}) \right]$$

$$= 2^{\frac{n}{2}} \left[ 2 \cos(\frac{n\pi}{4}) \right] = 2^{\frac{n}{2}+1} \left[ \cos(\frac{n\pi}{4}) \right]$$

**Example:** Given the following points in polar coordinates  $p_1(1, 45)$ ,  $p_2(2, -\frac{\pi}{3})$ , find the Cartesian representation for this points.

Sol:

$$p_1(1, 45), \quad r = 1, \quad \theta = 45$$

$$x = r \cos \theta = 1 \cos 45 = 1 * \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$y = r \sin \theta = 1 \sin 45 = 1 * \frac{1}{\sqrt{2}} = \frac{1}{\sqrt{2}}$$

$$\therefore z = \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}} i$$

$$p_2\left(2, -\frac{\pi}{3}\right), \quad r = 2, \quad \theta = -\frac{\pi}{3}$$

$$x = r \cos \theta = 2 \cos -\frac{\pi}{3} = 2 * \frac{1}{2} = 1$$

$$y = r \sin \theta = 2 \sin -\frac{\pi}{3} = -2 * 0.86 = -1.73$$

$$\therefore z = 1 - 1.73 i$$

**Not:**  $\sin(-x) = -\sin(x)$ ,  $\cos(-x) = \cos(x)$

**Example:** Change from polar to Cartesian coordinates.

Sol:

$$1) \quad r = \cos \theta + \sin \theta$$

$$x = r \cos \theta \rightarrow \cos \theta = \frac{x}{r}$$

$$y = r \sin \theta \rightarrow \sin \theta = \frac{y}{r}$$

$$\left( r = \frac{x}{r} + \frac{y}{r} \right) * r$$

$$r^2 = x + y$$

$$r^2 = x^2 + y^2 \quad (\text{From polar form}), \text{ then}$$

$$x + y = x^2 + y^2$$

$$(x^2 - x) + (y^2 - y) = 0 \quad (\text{Cartesian form})$$

$$2) \quad r^3 = 9$$

$$r \cdot r^2 = (x^2 + y^2)^{\frac{1}{2}} (x^2 + y^2) \rightarrow 9 = (x^2 + y^2)^{\frac{3}{2}} \quad (\text{Cartesian form})$$

$$3) \quad r \sin \theta = r \cos \theta \rightarrow x = y \quad (\text{Cartesian form})$$

$$4) \quad r = \frac{r \cos^2 \theta}{\sin \theta}$$

$$r \cos^2 \theta = r \sin \theta \rightarrow x^2 = y \quad (\text{Cartesian form})$$

**Example:** Change from cartesian to polar form.

$$1) x^2 + y^2 = 9$$

$$r^2 = x^2 + y^2 \rightarrow r^2 = 9 \rightarrow r = 3$$

$$2) \frac{x}{2} + \frac{y}{3} = 1$$

$$\frac{r \cos \theta}{2} + \frac{r \sin \theta}{3} = 1$$

$$\frac{3r \cos \theta + 2r \sin \theta}{6} = 1$$

$$3r \cos \theta + 2r \sin \theta = 6$$

$$r[3 \cos \theta + 2 \sin \theta] = 6$$

$$r = \frac{6}{3 \cos \theta + 2 \sin \theta}$$

### Euler's Formula:

$$z = x + iy$$

Cartesian form

$$z = r[\cos \theta + i \sin \theta]$$

Polar form

For any complex number we defined

$$e^{i\theta} = \cos \theta + i \sin \theta$$

$$e^{-i\theta} = \cos \theta - i \sin \theta$$

$$z = r e^{i\theta}$$

Exponention form

Where  $r = \sqrt{x^2 + y^2}$ ,  $\theta = \tan^{-1} \frac{y}{x}$

### Multiplication and Division of Exponention Form:

#### 1) Multiplication:

$$\begin{aligned} z_1 &= r_1 e^{i\theta_1}, & z_2 &= r_2 e^{i\theta_2} \\ z_1 z_2 &= r_1 r_2 e^{i(\theta_1 + \theta_2)} \end{aligned}$$

#### 2) Division:

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} e^{i(\theta_1 - \theta_2)}$$

#### 3) Power:

$$z^n = (r e^{i\theta})^n \rightarrow z^n = r^n e^{in\theta}$$

Example: If  $z = 2 + 2i$ , find (1) exp form      (2)polar form

Sol:

$$z = r e^{i\theta}$$

$$r = \sqrt{x^2 + y^2} = \sqrt{4 + 4} = \sqrt{8} = 2\sqrt{2}$$

$$\theta = \tan^{-1} \frac{y}{x} = \tan^{-1} \frac{2}{2} = \tan^{-1} 1 = \frac{\pi}{4}$$

$$z = 2\sqrt{2} e^{\frac{i\pi}{4}}$$

exp form

$$z = r[\cos \theta + i \sin \theta]$$

$$z = 2\sqrt{2} [\cos \frac{\pi}{4} + i \sin \frac{\pi}{4}]$$

polar form

**Note: For De-Moivre's' theorem when n is integer**

$$z^n = (x + iy)^n = r^n[\cos \theta + i \sin \theta]^n = r^n[\cos n\theta + i \sin n\theta]$$

**And when n is not integer**

$$z^{\frac{1}{n}} = r^{\frac{1}{n}} [\cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right)]$$

Where k is an integer 0, 1, 2... n-1

**Example:** Find the root of  $(1 + i)^{\frac{1}{3}}$

Sol:

$$z = 1 + i$$

$$r = \sqrt{1^2 + 1^2} = \sqrt{2}$$

$$\theta = \tan^{-1} \frac{1}{1} = \frac{\pi}{4}$$

$$\frac{1}{n} = \frac{1}{3} \rightarrow n = 3$$

$$\therefore k = 0, 1, 2$$

$$z^{\frac{1}{3}} = r^{\frac{1}{3}} \left[ \cos\left(\frac{\theta + 2\pi k}{n}\right) + i \sin\left(\frac{\theta + 2\pi k}{n}\right) \right]$$

For k=0

$$w_0 = (\sqrt{2})^{\frac{1}{3}} \left[ \cos\left(\frac{\frac{\pi}{4} + 2\pi \cdot 0}{3}\right) + i \sin\left(\frac{\frac{\pi}{4} + 2\pi \cdot 0}{3}\right) \right]$$

$$w_0 = 2^{\frac{1}{6}} \left[ \cos\frac{\pi}{12} + i \sin\frac{\pi}{12} \right]$$

K=1

$$w_1 = 2^{\frac{1}{6}} \left[ \cos\left(\frac{\frac{\pi}{4} + 2\pi}{3}\right) + i \sin\left(\frac{\frac{\pi}{4} + 2\pi}{3}\right) \right]$$

$$w_1 = 2^{\frac{1}{6}} \left[ \cos\left(\frac{\frac{\pi}{4} + 2\pi}{3}\right) + i \sin\left(\frac{\frac{\pi}{4} + 2\pi}{3}\right) \right]$$

$$w_1 = 2^{\frac{1}{6}} \left[ \cos\frac{3\pi}{4} + i \sin\frac{3\pi}{4} \right]$$

K=2

$$w_2 = 2^{\frac{1}{6}} \left[ \cos\left(\frac{\frac{\pi}{4} + 2\pi \cdot 2}{3}\right) + i \sin\left(\frac{\frac{\pi}{4} + 2\pi \cdot 2}{3}\right) \right]$$

$$w_2 = 2^{\frac{1}{6}} \left[ \cos \frac{17\pi}{12} + i \sin \frac{17\pi}{12} \right]$$

**Example:** Prove that  $e^{i\theta} = \cos \theta + i \sin \theta$

Sol:

$$\cos \theta = \left( \frac{e^{i\theta} + e^{-i\theta}}{2} \right), \quad \sin \theta = \left( \frac{e^{i\theta} - e^{-i\theta}}{2i} \right)$$

$$\cos \theta + i \sin \theta = \frac{e^{i\theta} + e^{-i\theta}}{2} + i \frac{e^{i\theta} - e^{-i\theta}}{2i}$$

$$\cos \theta + i \sin \theta = \frac{e^{i\theta} + e^{-i\theta} + e^{i\theta} - e^{-i\theta}}{2} = \frac{2e^{i\theta}}{2} = e^{i\theta}$$

# Homework

1) Solve the following equation for the real numbers  $x$  and  $y$ .

$$(3 + 4i)^2 - 2(x - iy) = x + iy$$

2) Find the four forth roots of (-16)

3) Change from Cartesian to polar coordinate     $x^2 + y^2 - 6x = 0$

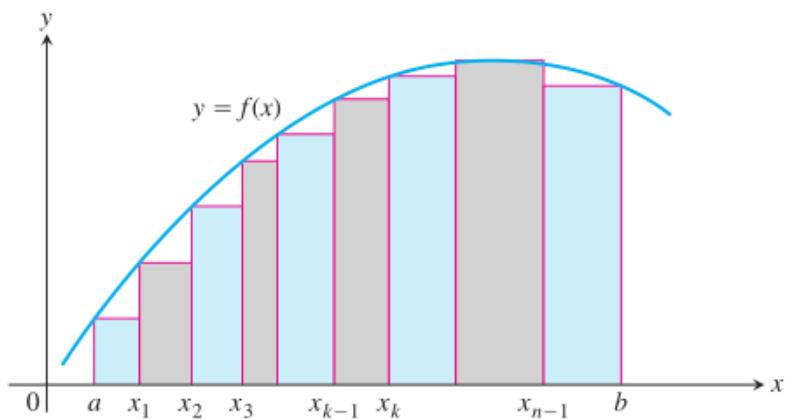
4) Find  $(\sqrt{3} - i)^{10}$

5) Show that  $|\bar{z}| = |z|$

# Chapter Three

## Multiple Integrals

If  $f(x) > 0$  then  $\int_a^b f(x) dx$  represent the area under the curve from  $a = x_1$  to  $b = x_{n-1}$



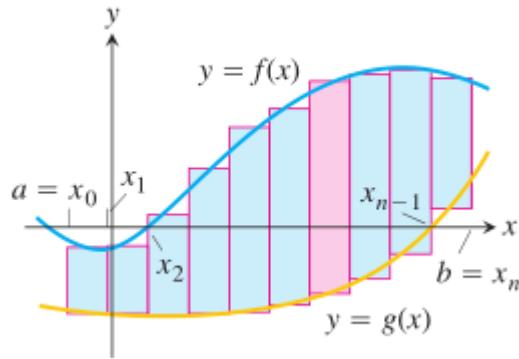
Then the area between curves is given by the formulas:

$$1) \ A = \int_a^b [f(x) - g(x)] dx$$

For a region bounded by  $y = f(x)$  and  $y = g(x)$  on the left and on the right by  $x = a$  and  $x = b$ .

$$2) \quad A = \int_c^d [f(y) - g(y)] dy$$

For a region bounded by  $x = f(y)$  and  $x = g(y)$  on the left and on the right by  $y = c$  and  $y = d$ .



**Example:** Find the area between  $y = x$  and  $y = x^2$  from  $x = 0$  to  $x = 1$

Sol:

$$A = \int_0^1 |x - x^2| dx = \int_0^1 (x - x^2) dx = \frac{1}{2} x^2 - \frac{x^3}{3} \Big|_0^1 = \frac{1}{6}$$

**Example:** Calculate the area between  $x = y + 3$ ,  $x = y^2$  from

$$y = -1 \text{ to } y = 1.$$

Sol:

$$\begin{aligned}
 A &= \int_{-1}^1 |y + 3 - y^2| \, dy = \int_{-1}^1 (y + 3 - y^2) \, dy \\
 A &= \left[ \frac{y^2}{2} + 3y - \frac{y^3}{3} \right]_{-1}^1 = \left( \frac{1}{2} + 3 - \frac{1}{3} \right) - \left( \frac{1}{2} - 3 + \frac{1}{3} \right) \\
 &= \left( \frac{3+18-2}{6} \right) - \left( \frac{3-18+2}{6} \right) = \frac{19}{6} - \frac{-13}{6} = \frac{32}{6}
 \end{aligned}$$

## Double Integrals of Rectangles:

Suppose that  $f(x, y)$  is defined on rectangular region  $R$  defined by:

$$R: a \leq x \leq b, \quad c \leq y \leq d$$

We imagine  $R$  to be covered by a network of lines parallel to the  $x$  and  $y$  axis.

These lines divide  $R$  into small pieces of area.

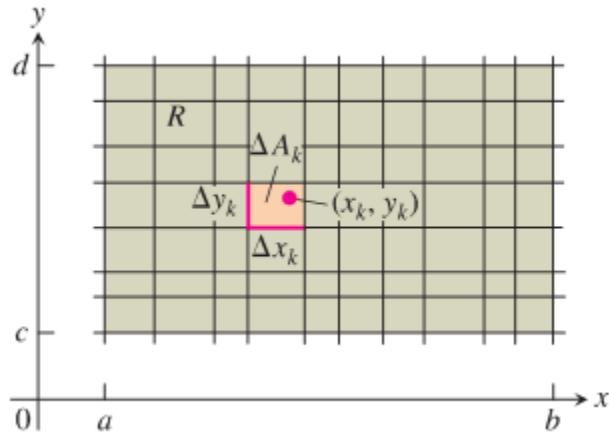
$$\Delta A = \Delta x \Delta y$$

If  $f$  is continuous through  $R$ , then

$$\iint_A f(x, y) dA \quad \text{or} \quad \iint_R f(x, y) dx dy$$

Thus:

$$\iint_A f(x, y) dA = \lim_{\nabla A \rightarrow 0} \sum f(x, y) dA$$



### Properties of the Double Integral:

- 1)  $\iint_A k f(x, y) dA = k \iint_R f(x, y) dA$  ,  $k$  is constant.
- 2)  $\iint_R [f(x, y) dA \pm g(x, y)] dA = \iint_R f(x, y) dA \pm \iint_R g(x, y) dA$
- 3)  $\iint_R f(x, y) dA \geq 0$  if  $f(x, y) \geq 0$  on  $R$
- 4)  $\iint_R f(x, y) dA \geq \iint_R g(x, y) dA$  if  $f(x, y) \geq g(x, y)$ .
- 5) If  $R = R_1 \cup R_2$  with  $R_1 \cap R_2 = \emptyset$  then

$$\iint_R f(x, y) dA = \iint_{R_1 \cup R_2} f(x, y) dA = \iint_{R_1} f(x, y) dA + \iint_{R_2} f(x, y) dA$$

**Fubini's Theorem (First Form):**

If  $f(x,y)$  is continuous function on the rectangular region

$$R = \{(x,y) : a \leq x \leq b, c \leq y \leq d\}$$

$$\iint_R f(x,y) dA = \int_a^b \int_c^d f(x,y) dy dx = \int_c^d \int_a^b f(x,y) dx dy$$

**Example:** Let  $f(x,y) = 1 - 6x^2 y$  be a function defined on the region

$$R = \{(x,y) : 0 \leq x \leq 2, -1 \leq y \leq 1\} \text{ calculate } \iint_R f(x,y) dA.$$

Sol:

$$\begin{aligned} \iint_R f(x,y) dA &= \int_{x=0}^2 \int_{y=-1}^1 (1 - 6x^2 y) dy dx = \int_0^2 [y - \frac{6}{2} x^2 y^2]_{-1}^1 dx \\ &= \int_0^2 [(1 - 3x^2) - (-1 - 3x^2)] dx = 2 \int_0^2 dx = [2x]_0^2 \\ &= 2(2 - 0) = 4 \end{aligned}$$

**Example:** Evaluate:  $\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y) dx dy$

Sol:

$$\int_{\pi}^{2\pi} \int_0^{\pi} (\sin x + \cos y) dx dy = \int_{\pi}^{2\pi} [-\cos x + x \cos y]_0^{\pi} dy$$

$$\begin{aligned}
 &= \int_{\pi}^{2\pi} [(-\cos \pi + \pi \cos y) - (-\cos 0 + 0)] dy \\
 &= \int_{\pi}^{2\pi} (2 + \pi \cos y) dy = [2y + \pi \sin y]_{\pi}^{2\pi} \\
 &= (4\pi + 0) - (2\pi + 0) = 2\pi
 \end{aligned}$$

**Example:** Find the volume under the plane  $z = 4 - x - y$  over the region R in x-y plane where:  $R = \{(x, y): 0 \leq x \leq 2, 0 \leq y \leq 1\}$ .

Sol:

$$\begin{aligned}
 v &= \iint_R f(x, y) dA = \int_0^2 \int_0^1 (4 - x - y) dy dx \\
 v &= \int_0^2 \left[ 4y - xy - \frac{1}{2}y^2 \right]_0^1 dx = \int_0^2 \left( 4 - x - \frac{1}{2} \right) dx \\
 v &= \int_0^2 \left( \frac{7}{2} - x \right) dx = \left[ \frac{7}{2}x - \frac{1}{2}x^2 \right]_0^2 \\
 v &= (7 - 2) - 0 = 5
 \end{aligned}$$

## Double Integral Over Bounded Non-Rectangular Region:

### Fubini's Theorem (Second Form):

Let  $f(x, y)$  be a continuous on region  $R$

- 1) If  $R$  defined by  $a \leq x \leq b, f_1(x) \leq y \leq f_2(x)$  with  $f_1$  and  $f_2$  on  $[a, b]$ , then

$$\iint_R f(x, y) dA = \int_a^b \int_{f_1(x)}^{f_2(x)} f(x, y) dy dx$$

- 2) If  $R$  defined by  $c \leq y \leq d, g_1(y) \leq x \leq g_2(y)$  with  $g_1$  and  $g_2$  on  $[c, d]$ , then

$$\iint_R f(x, y) dA = \int_c^d \int_{g_1(y)}^{g_2(y)} f(x, y) dx dy$$

**Example:** Find the integral of the function  $f(x, y) = x^2 y$  over the region  $R$ , where:  $R = \{(x, y): 2y^2 \leq x \leq 2y, 0 \leq y \leq 1\}$ .

Sol:

$$\iint_R f(x, y) dA = \int_{y=0}^1 \int_{x=2y^2}^{2y} x^2 y dx dy$$

$$= \int_0^1 \left[ \frac{1}{3} x^3 y \right]_{2y^2}^{2y} dy = \int_0^1 \frac{y}{3} (8y^3 - 8y^6) dy$$

$$= \frac{8}{3} \int_0^1 (y^4 - y^7) dy = \left[ \frac{y^5}{5} - \frac{y^8}{8} \right]_0^1 = \frac{8}{3} \left( \frac{1}{5} - \frac{1}{8} \right) = \frac{1}{5}$$

**Example:** Evaluate  $\int_{y=0}^{\ln 8} \int_{x=0}^{1-y} e^{x+y} dx dy$

Sol:

$$\begin{aligned} \int_0^{\ln 8} [e^x (e^y)]_0^{1-y} dy &= \int_0^{\ln 8} e^y (e^{1-y} - e^0) dy = [e^y - e^y]_0^{\ln 8} \\ &= e \ln 8 - 8 + 1 = e \ln 8 - 7 \end{aligned}$$

**Example:** Find the volume of the solid whose base is the region in xy-plane

Bounded by the parabola  $y = 2 - x^2$  and the line  $y = x$  while the solid is bounded by the plane  $z = 3 - x$ .

Sol:

$$2 - x^2 \leq y \leq x,$$

$$y = 2 - x^2, \quad y = x$$

$$x = 2 - x^2 \rightarrow x^2 + x - 2 = 0$$

$$(x+2)(x-1) = 0 \rightarrow x = 1, x = -2$$

$$v = \iint_R f(x, y) dA = \int_{x=-2}^1 \int_{y=x}^{2-x^2} (3 - x) dy dx$$

$$\begin{aligned}
 & \int_{-2}^1 [(3-x)y]_x^{2-x^2} dx = \int_{-2}^1 (3-x)(2-x^2-x)dx \\
 &= \int_{-2}^1 (6 - 3x^2 - 3x - 2x + x^3 + x^2)dx \\
 &= \int_{-2}^1 (x^3 - 2x^2 - 5x + 6) dx \\
 &= \left[ \frac{1}{4}x^4 - \frac{2}{3}x^3 - \frac{5}{2}x^2 + 6x \right]_{-2}^1 \\
 &= \left( \frac{1}{4} - \frac{2}{3} - \frac{5}{2} + 6 \right) - \left( 4 + \frac{16}{3} - 10 - 12 \right) = \frac{54}{3}
 \end{aligned}$$

**Example:** Find the area of the region R in the first quadrant bounded by  $y = x$  and  $y = x^2$ .

Sol:

$$x \leq y \leq x^2,$$

$$y = x^2, \quad y = x$$

$$x = x^2 \rightarrow x^2 - x = 0$$

$$x(x-1) = 0 \rightarrow x = 0, x = 1$$

$$0 \leq x \leq 1$$

$$A = \iint_R dA = \int_{x=0}^1 \int_{x^2}^x dy dx$$

$$\begin{aligned}
 A &= \int_0^1 [y]_{x^2}^{x^2} dx = \int_0^1 (x - x^2) dx = \left[ \frac{1}{2}x^2 - \frac{1}{3}x^3 \right]_0^1 \\
 &= \frac{1}{2} - \frac{1}{3} = \frac{1}{6}
 \end{aligned}$$

**Example:** Find the area of the region R enclosed by the line  $y = x + 2$  and the parabola  $y = x^2$ .

Sol:

$$y = x^2, \quad y = x + 2$$

$$x + 2 = x^2 \rightarrow x^2 - x - 2 = 0$$

$$(x - 2)(x + 1) = 0 \rightarrow x = -1, x = 2$$

$$-1 \leq x \leq 2$$

$$A = \iint_R dA = \int_{-1}^2 \int_{x^2}^{x+2} dy dx$$

$$A = \int_{-1}^2 [y]_{x^2}^{x+2} dx = \int_{-1}^2 (x + 2 - x^2) dx = \left[ \frac{1}{2}x^2 + 2x - \frac{1}{3}x^3 \right]_{-1}^2$$

$$A = \left( 2 + 4 - \frac{8}{3} \right) - \left( \frac{1}{2} - 2 + \frac{1}{3} \right) = \frac{9}{2}$$

# Homework

- Evaluate the following integrals:

$$1) \int_0^2 \int_{y/2}^1 e^x dx dy$$

$$2) \int_0^{2\pi} \int_{\ln x}^0 dy dx$$

$$3) \int_{-1}^0 \int_{-x}^{x+1} (x^2 + y - 2) dy dx$$

$$4) \int_1^3 \int_{1-y}^{y-1} (2x - y^2) dx dy$$

- 5) Find the volume of the region bounded above by the elliptical paraboloid  $z = 4x + x^2 - y^2$  and below by the square  $R: -1 \leq x \leq 1, 0 \leq y \leq 1$

# Chapter Four

## Complex Functions

### 1) Trigonometric Functions:

$$(a) \sin z = \frac{e^{iz} - e^{-iz}}{2i}$$

$$(b) \cos z = \frac{e^{iz} + e^{-iz}}{2}$$

$$(c) \tan z = \frac{e^{iz} - e^{-iz}}{i(e^{iz} + e^{-iz})}$$

$$(d) \cot z = \frac{i(e^{iz} + e^{-iz})}{(e^{iz} - e^{-iz})}$$

$$(e) \sec z = \frac{2}{e^{iz} + e^{-iz}}$$

$$(f) \csc z = \frac{2i}{e^{iz} - e^{-iz}}$$

### 2) Hyperbolic Functions:

$$(a) \sinh z = \frac{e^z - e^{-z}}{2}$$

$$(b) \cosh z = \frac{e^z + e^{-z}}{2}$$

$$(c) \tanh z = \frac{e^z - e^{-z}}{(e^z + e^{-z})}$$

$$(d) \coth z = \frac{(e^z + e^{-z})}{(e^z - e^{-z})}$$

$$(e) \operatorname{sech} z = \frac{2}{e^z + e^{-z}}$$

$$(f) \operatorname{csch} z = \frac{2}{e^z - e^{-z}}$$

Example: Prove that:  $\cos z = \cos x \cosh y - i \sin x \sinh y$

Sol:

We take the right side

$$\begin{aligned}
 &= \left[ \frac{e^{ix} + e^{-ix}}{2} \right] \left[ \frac{e^y + e^{-y}}{2} \right] - i \left[ \frac{e^{ix} - e^{-ix}}{2i} \right] \left[ \frac{e^y - e^{-y}}{2} \right] \\
 &= \frac{e^{ix+y} + e^{ix-y} + e^{-ix+y} + e^{-ix-y} - (e^{ix+y} - e^{ix-y} - e^{-ix+y} + e^{-ix-y})}{4} \\
 &= \frac{e^{ix+y} + e^{ix-y} + e^{-ix+y} + e^{-ix-y} - e^{ix+y} + e^{ix-y} + e^{-ix+y} - e^{-ix-y}}{4}
 \end{aligned}$$

$$\begin{aligned}
 &= \frac{2e^{ix-y} + 2e^{-ix+y}}{4} = \frac{e^{-(y-ix)} + e^{-i(ix-y)}}{2} \\
 &= \frac{e^{i^2(y-ix)} + e^{i^2(ix-y)}}{2} = \frac{e^{i(iy+x)} + e^{i(-x-iy)}}{2} = \frac{e^{i(x+iy)} + e^{-i(x+iy)}}{2} \\
 &= \frac{e^{iz} + e^{-iz}}{2} = \cos z
 \end{aligned}$$

**Example:** Proof that:  $\sin(iz) = i \sinh(z)$

Sol:

$$\begin{aligned}
 \sin(iz) &= \frac{e^{i(iz)} - e^{-i(iz)}}{2i} = \frac{e^{-z} - e^z}{2i} \\
 &= -\frac{1}{i} \left( \frac{e^z - e^{-z}}{2} \right) = i \left( \frac{e^z - e^{-z}}{2i} \right) = i \sinh(z)
 \end{aligned}$$

### Expansion of Trigonometric Functions:

Let  $r=1$  therefore the De-mover's formula begin:

$$z = \cos \theta + i \sin \theta \quad \text{and} \quad \frac{1}{z} = \cos \theta - i \sin \theta$$

$$z + \frac{1}{z} = 2 \cos \theta, \quad z - \frac{1}{z} = 2i \sin \theta$$

$$z^n + \frac{1}{z^n} = 2 \cos n\theta, \quad z^n - \frac{1}{z^n} = 2i \sin n\theta$$

**Example:** Find the expansion for  $\cos^3 \theta$ ?

Sol:

$$z + \frac{1}{z} = 2 \cos \theta \rightarrow \cos \theta = \frac{1}{2} \left( z + \frac{1}{z} \right)$$

$$\cos^3 \theta = \left[ \frac{1}{2} \left( z + \frac{1}{z} \right) \right]^3 = \frac{1}{8} \left( z^3 + 3z^2 \cdot \frac{1}{z} + \frac{6}{2!} z \cdot \frac{1}{z^2} + \frac{1}{z^3} \right)$$

$$\cos^3 \theta = \frac{1}{8} \left[ \left( z^3 + \frac{1}{z^3} \right) + 3 \left( z + \frac{1}{z} \right) \right]$$

$$\cos^3 \theta = \frac{1}{8} [2 \cos 3\theta + 3(2\cos \theta)] = \frac{\cos 3\theta + 3 \cos \theta}{4}$$

## Inverse of Trigonometric and Hyperbolic Functions:

If :  $z = \sin w \rightarrow w = \sin^{-1} z$

$$1) \sin^{-1} z = \frac{1}{i} \ln(iz \pm \sqrt{1 - z^2})$$

$$2) \cos^{-1} z = \frac{1}{i} \ln(z \pm \sqrt{z^2 - 1})$$

$$3) \tan^{-1} z = \frac{1}{2i} \ln\left(\frac{1+iz}{1-iz}\right)$$

$$4) \sinh^{-1} z = \ln(z + \sqrt{z^2 + 1})$$

$$5) \cosh^{-1} z = \ln(z + \sqrt{z^2 - 1})$$

$$6) \tanh^{-1} z = \frac{1}{2} \ln\left(\frac{1+z}{1-z}\right)$$

**Example:** Prove that  $\sin^{-1} z = \frac{1}{i} \ln(iz \pm \sqrt{1 - z^2})$

Sol:

$$\text{Let } w = \sin^{-1} z \rightarrow z = \sin w = \frac{e^{iw} - e^{-iw}}{2i}$$

$$2iz = e^{iw} - e^{-iw} \rightarrow (e^{iw} - e^{-iw} - 2iz = 0) * e^{iw}$$

$$e^{2iw} - 2iz e^{iw} - 1 = 0 \rightarrow (e^{iw})^2 - 2iz e^{iw} - 1 = 0$$

$$e^{iw} = \frac{2iz \pm \sqrt{-4z^2 + 4}}{2} = \frac{2iz \pm 2\sqrt{-z^2 + 1}}{2}$$

$$e^{iw} = \frac{2(iz \pm \sqrt{1-z^2})}{2} = iz \pm \sqrt{1-z^2}$$

$$iw = \ln(iz \pm \sqrt{1-z^2}) \rightarrow w = \frac{1}{i} \ln(iz \pm \sqrt{1-z^2})$$

$$\therefore w = \sin^{-1} z$$

### Cauchy – Rieman Equations:

Let  $f(z) = u(x, y) + i v(x, y)$

The necessary condition that  $f(z)$  be analytic in region R is that

$u(x, y)$  and  $v(x, y)$  satisfy:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} \quad \text{and} \quad \frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad (\text{Cauchy's condition})$$

**Example:** Check if the function  $f(z)$  is analytic when  $f(z) = x - iy$

Sol:

$$f(z) = x - iy$$

$$u = x, \quad v = -y$$

$$\frac{\partial u}{\partial x} = 1, \quad \frac{\partial v}{\partial y} = -1$$

$$\because \frac{\partial u}{\partial x} \neq \frac{\partial v}{\partial y} \quad \text{then } f(z) \text{ is not analytic}$$

**Example:** If  $f(z) = z^2$  then is analytic or no?

Sol:

$$f(z) = z^2 = (x + iy)^2 = x^2 + 2ixy - y^2 = (x^2 - y^2) + 2ixy$$

$$u = x^2 - y^2, \quad v = 2xy$$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial y} = 2x$$

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial y} = 2x \quad \text{first condition}$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial x} = 2y$$

$$\frac{\partial u}{\partial y} = -\frac{\partial v}{\partial x} \quad \text{second condition}$$

Then  $f(z)$  is analytic.

**Example:** Show that  $\frac{d}{dz} (\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}$

Sol:

$$\text{Let } \sin^{-1} z = w \rightarrow z = \sin w \rightarrow z^2 = \sin^2 w$$

$$\frac{d}{dz} (z) = \frac{d}{dz} (\sin w) \rightarrow \frac{dz}{dz} = \cos w \frac{dw}{dz}$$

$$1 = \cos w \frac{dw}{dz} \rightarrow \frac{dw}{dz} = \frac{1}{\cos w}$$

$$\cos^2 w + \sin^2 w = 1 \rightarrow \cos^2 w + \sin^2 w = 1$$

$$\cos^2 w = 1 - \sin^2 w$$

$$\cos w = \sqrt{1 - \sin^2 w} = \sqrt{1 - z^2}$$

$$\frac{d w}{d z} = \frac{1}{\sqrt{1-z^2}} \quad \rightarrow \quad \frac{d}{d z} (\sin^{-1} z) = \frac{1}{\sqrt{1-z^2}}$$

# Homework

1) Are the following functions analytic?

a)  $f(z) = z \bar{z}$

b)  $f(z) = z + z^2$

2) Prove that:  $\frac{d}{dz} (\tan^{-1} z) = \frac{1}{1+z^2}$

3) Find the expansion of  $\sin^3 \theta$

4) Proof that:  $\cosh iz = \cos z$

5) Prove that:  $\cosh^2 z - \sinh^2 z = 1$

6) Prove that:  $\cosh^2 z + \sinh^2 z = \cosh 2z$

## Chapter Five

### Ordinary Differential Equations

**Definition:** A differential equation is an equation involving an unknown function and its derivatives.

#### Order and Degree:

**The order:** is the highest derivative appearing in the equation.

**The degree:** is the power to which the highest order derivative is raised.

#### First Order of Ordinary Differential Equations (ODEs):

There are four ways to solve the First order of ordinary differential equations:

- 1) Separation variable.
- 2) Homogenous differential equations.
- 3) Exact differential equations.
- 4) Integrating factor of the linear equations.

#### **1) Separation Variable:**

If  $f(x, y) = \frac{dy}{dx}$  is said to be separable if it can be written in the form:

$$\frac{dy}{dx} = f(x) g(y)$$

**Example:** Solve the equation  $\frac{dy}{dx} = 1 + y^2$

Sol:

$$\frac{dy}{1+y^2} = dx$$

$$\int \frac{dy}{1+y^2} = \int dx \rightarrow \tan^{-1} y = x + c$$

$$y = \tan(x + c)$$

**Example:** Find the solution of  $y' + (x + 2)y^2 = 0$

Sol:

$$\frac{dy}{dx} + (x + 2)y^2 = 0$$

$$\frac{dy}{dx} = -(x + 2)y^2$$

$$\frac{dy}{y^2} = -(x + 2)dx$$

$$-\int \frac{dy}{y^2} = \int (x + 2)dx$$

$$\frac{1}{y} = \frac{x^2}{2} + 2x + c$$

$$y = \frac{1}{\frac{x^2}{2} + 2x + c}$$

**Example:** Solve the equation  $y' = k y$

Sol:

$$\frac{dy}{dx} = k y$$

$$\frac{dy}{y} = k dx$$

$$\int \frac{dy}{y} = k \int dx$$

$$\ln y = k x + c$$

$$y = e^{kx} + e^c$$

## 2) Homogenous Equations:

$p(x, y)dx + q(x, y)dy = 0$  is called homogenous if p and q are homogenous functions of same degree when written in the form  $\frac{dy}{dx} = f(x, y) = f(u)$

The function  $f(x, y)$  can be written as:  $f(x, y) = g(\frac{y}{x})$

We substitute u instead of  $(\frac{y}{x})$  and then find the solution of this differential equation as:

$$\int \frac{du}{f(u) - u} = \int \frac{dx}{x}$$

Where:  $u = \frac{y}{x}$

Example: Solve the equation  $\dot{y} = \frac{4x^2+y^2}{xy}$

Sol:

$$\dot{y} = f\left(\frac{y}{x}\right) = \frac{4x^2}{xy} + \frac{y^2}{xy} = u \left(\frac{x}{y}\right) + \frac{y}{x}$$

$$\text{Let } u = \frac{y}{x} \text{ then } f(u) = \frac{4}{u} + u$$

General solution:

$$\int \frac{du}{f(u)-u} = \int \frac{dx}{x}$$

$$\int \frac{du}{\left(\frac{4}{u}+u\right)-u} = \int \frac{dx}{x}$$

$$\frac{1}{4} \int u \, du = \int \frac{dx}{x} \rightarrow \frac{1}{8} u^2 = \ln x + c$$

$$u^2 = 8 \ln x + 8c \rightarrow \left(\frac{y}{x}\right)^2 = 8 \ln x + 8c$$

$$y^2 = x^2(8 \ln x + 8c) \rightarrow y = x \sqrt{8 \ln x + 8c}$$

Example: Solve  $\dot{y}e^{\frac{y}{x}} = 2 \left[ e^{\frac{y}{x}} - 1 \right] + \frac{y}{x} e^{\frac{y}{x}}$

Sol:

We multiplies both sides by  $e^{-\frac{y}{x}}$

$$\dot{y} e^{\frac{y}{x}} = 2(e^{\frac{y}{x}} - 1) + \frac{y}{x} e^{\frac{y}{x}} * e^{-\frac{y}{x}}$$

$$y' = 2(e^{\frac{y}{x}} - 1) + \frac{y}{x} \quad (u = \frac{y}{x})$$

$$F(u) = 2(1 - e^{-u}) + u$$

$$\int \frac{du}{f(u)-u} = \int \frac{dx}{x}$$

$$\int \frac{du}{2(1-e^{-u})+u-u} = \int \frac{dx}{x}$$

$$\frac{1}{2} \ln(1 - e^{-u}) = \ln x + c$$

$$\frac{1}{2} \ln \left( 1 - e^{-\frac{y}{x}} \right) = \ln x + c$$

### 3) Exact Equations:

The first order differential equation

$$M(x, y)dx + N(x, y)dy = 0$$

Said to be exact if a function  $f(x, y)$  exists such that the total differential.

$$d[f(x, y)] = M(x, y)dx + N(x, y)dy$$

It follows directly that if

$$M(x, y)dx + N(x, y)dy = 0 \quad (*)$$

Is exact, the  $d[f(x, y)] = 0$ , so the general solution of  $(*)$  is must be constant.

### Steps for Solving an Exact Differential Equation:

- 1) Match the equation to the form  $M(x,y)dx + N(x,y) dy = 0$  to identify M and N.
- 2) Integrate M (or N) with respect to x (or y), writing the constant of integration as g(x) or g(y).
- 3) Differentiate with respect to y (or x) and set the result equal to N (or M) to  $\dot{g}(x)$  or  $\dot{g}(y)$ .
- 4) Integrate to g(y) or g (x ).
- 5) Write the solution of the exact equation as  $f(x,y) = c$

**Example:** Solve the equation  $2xy dx + (x^2 + 1)dy = 0$

Sol:

$$M = 2xy = \frac{\partial v}{\partial x}, \quad \frac{\partial M}{\partial y} = 2x$$

$$N = x^2 + 1 = \frac{\partial v}{\partial y}, \quad , \quad \frac{\partial N}{\partial x} = 2x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad so \ it \ is \ exact.$$

$$\frac{\partial v}{\partial x} = 2xy \rightarrow v = \int 2xy \, dx$$

$$v = x^2 y + A(y) \quad (*)$$

$$\frac{\partial v}{\partial y} = x^2 + A(y) = N$$

$$\therefore x^2 + A(y) = x^2 + 1$$

$$A(y) = 1 \rightarrow \int A(y) = \int dy$$

$$A(y) = y + c$$

**Example:** Solve the equation  $e^x(y dx + dy) = 0$

Sol:

$$ye^x dx + e^x dy = 0$$

$$M = ye^x = \frac{\partial v}{\partial x}, \quad \frac{\partial M}{\partial y} = e^x$$

$$N = e^x = \frac{\partial v}{\partial y}, \quad \frac{\partial N}{\partial x} = e^x$$

$$\therefore \frac{\partial M}{\partial y} = \frac{\partial N}{\partial x} \quad \text{so it is exact.}$$

$$\frac{\partial v}{\partial x} = e^x y \rightarrow v = \int e^x y dx$$

$$v = e^x y + A(y) \quad (*)$$

$$\frac{\partial v}{\partial y} = e^x + A(y) = N$$

$$\therefore e^x + A(y) = e^x$$

$$A = 0 \rightarrow \int A(y) = 0$$

$$A(y) = c$$

#### 4- Integrating Factor:

The standard form of the linear differential equation is:

$$\frac{dy}{dx} + P(x)y = Q(x) \dots (*)$$

Where  $P(x)$  and  $Q(x)$  known functions and the integrating factor of the linear equation are is:

$$I = e^{\int p(x)dx} \quad \text{the general solution of } (*)$$

$$I \cdot y(x) = \int I Q(x) dx$$

$$e^{\int p(x)dx} y(x) = \int e^{\int p(x)dx} Q(x) dx$$

**Example:** Solve the equation  $\frac{dy}{dx} - y \tan x = \sin x \cos^2 x$

Sol:

$$P(x) = -\tan x, \quad Q(x) = \sin x \cos^2 x$$

$$I = e^{\int p(x)dx} = e^{-\int \tan(x)dx} = e^{-\int \frac{\sin(x)}{\cos(x)} dx}$$

$$e^{\ln(\cos x)} = \cos x$$

$$y \cos x = \int \cos(x) \sin(x) \cos^2 y dx$$

$$y \cos x = \int \sin(x) \cos^3 y dx$$

$$y \cos x = -\frac{1}{4} \cos^4 x + c$$

$$y = -\frac{1}{4} \cos^3 x + \frac{c}{\cos x}$$

**Example:** Solve the equation:  $\left(\frac{1}{x^2} + \frac{2y}{x^3}\right) dx - \frac{1}{x^2} dy = 0$

Sol:

$$\left(\frac{1}{x^2} + \frac{2y}{x^3}\right) dx = \frac{1}{x^2} dy$$

$$\left(\frac{1}{x^2} + \frac{2y}{x^3}\right) = \frac{1}{x^2} \frac{dy}{dx} \rightarrow \left[\frac{1}{x^2} \frac{dy}{dx} - \left(\frac{1}{x^2} + \frac{2y}{x^3}\right)\right] * x^2 = 0$$

$$\frac{dy}{dx} - 1 - \frac{2y}{x} = 0 \rightarrow \frac{dy}{dx} - \frac{2y}{x} = 1$$

$$P(x) = -\frac{2}{x}, \quad Q(x) = 1$$

$$I = e^{\int p(x)dx} = e^{-2 \int \frac{dx}{x}} = e^{-2 \ln x}$$

$$I = e^{\ln x^{-2}} = x^{-2} = \frac{1}{x^2}$$

$$\therefore \frac{y}{x^2} = \int \frac{1}{x^2} dx \rightarrow \frac{y}{x^2} = -\frac{1}{x} + c$$

$$\begin{aligned}\therefore \frac{y}{x^2} &= \int \frac{1}{x^2} dx \rightarrow \frac{y}{x^2} = -\frac{1}{x} + c \\ \therefore y &= cx^2 - x\end{aligned}$$

**Example:** Solve  $(\frac{dy}{dx} + y \cot x) = \sin 2x$

Sol:

$$\frac{dy}{dx} + y \cot(x) = \sin(2x)$$

$$p(x) = \cot x \quad , \quad Q(x) = \sin(2x)$$

$$I = e^{\int \cot x dx} = e^{\int \frac{\cos x dx}{\sin x}} = e^{\ln(\sin x)} = \sin x$$

$$y \sin x = \int \sin x \sin(2x) dx = 2 \int \sin^2 x \cos x dx$$

$$y \sin x = \frac{2}{3} \sin^3 x + c$$

$$y = \frac{2}{3} \frac{\sin^3 x}{\sin x} + \frac{c}{\sin x}$$

$$y = \frac{2}{3} \sin^2 x + \frac{c}{\sin x}$$

**Note that:**  $\sin(2x) = 2 \sin x \cos x$

# Homework

-Solve the following differential equations:

$$1) \frac{dy}{dx} + \frac{y}{x} = e^x \quad (\text{Integrating factor}).$$

$$2) (2x + \sin x)dx + x \cos y \ dy = 0 \quad (\text{Exact}).$$

$$3) \frac{dy}{dx} - 2y + a = 0 \quad (\text{Separation variables}).$$

$$4) \frac{dy}{dx} - \frac{3y}{x} = x^2 \quad (\text{Integrating factor}).$$

$$5) x \ dx + x(1 + y)dy = 0 \quad (\text{Exact}).$$

$$6) x \frac{dy}{dx} = m y^2 \quad (\text{Separation variables}).$$



## Chapter Six

### Sequences and Series

#### Sequences:

A sequence of numbers is a function whose domain is the set of positive integers.

#### Example:

0 , 1, 2, ..., n-1      for a sequence whose defining rule is  $a_n = n - 1$

1,  $\frac{1}{2}$ ,  $\frac{1}{3}$ ,  $\frac{1}{4}$ , ...,  $\frac{1}{n}$       for a sequence whose defining rule is  $a_n = \frac{1}{n}$

The index n is the domain of the sequence. While the numbers in the range      of the sequence are called the terms of the sequence, and the number  $a_n$  being called the  $n^{\text{th}}$  term, or the term with index.

#### Example:

$a_n = \frac{n+1}{n}$       then the terms are:

$$a_1 = 2, \quad a_2 = \frac{3}{2}, \quad a_3 = \frac{4}{3}, \dots, \frac{n+1}{n}$$

Example: Find the first five terms of the following

a)  $\frac{2n-1}{3n+2}$

b)  $\frac{1-(-1)^n}{n^3}$

Sol:

a)  $\frac{1}{5}, \frac{3}{8}, \frac{5}{11}, \frac{7}{14}, \frac{9}{17}$

b)  $2, 0, \frac{2}{27}, 0, \frac{2}{125}$

## Infinite Series:

Infinite series are sequences of a special kind: those in which the  $n^{\text{th}}$ -term is the sum of the first  $n$  terms of a related sequence.

**Example:** Suppose that we start with the sequence

$$1, \frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \frac{1}{16}, \frac{1}{32}, \frac{1}{64}, \dots$$

Sol:

$$s_1 = a_1 = 1$$

$$s_2 = a_1 + a_2 = 1 + \frac{1}{2} = \frac{3}{2}$$

$$s_3 = a_1 + a_2 + a_3 = 1 + \frac{1}{2} + \frac{1}{4} = \frac{7}{4}$$

As the first three terms of the sequence  $\{s_n\}$ .

When the sequence  $\{s_n\}$  is formed in this way from a given sequence  $\{a_n\}$  by the rule:  $s_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k$

The result is called an infinite series.

## Geometric Series:

A series of the form  $a + ar + ar^2 + ar^3 + \dots, ar^{n-1} + \dots$

is called a geometric series. The ratio of any term to the one before it is if  $r < 1$ .

the geometric series converges  $\frac{a}{1-r}$ , if  $|r| \geq 1$ , the series diverges unless

$a = 0$ .

If  $a = 0$ , the series converges to 0.

Example: Geometric series with  $a = \frac{1}{9}$  and  $r = \frac{1}{3}$

Sol:

$$\frac{1}{9} + \frac{1}{27} + \frac{1}{81} + \dots = \frac{1}{9} \left( \frac{1}{3} + \frac{1}{3^2} + \dots \right) = \left( \frac{\frac{1}{9}}{1 - \frac{1}{3}} \right) = \frac{1}{6}$$

Geometric series with  $a = 4$  and  $r = -\frac{1}{2}$

$$4 - 2 + 1 - \frac{1}{2} + \frac{1}{4} - \dots = 4 \left( 1 - \frac{1}{2} + \frac{1}{4} - \frac{1}{8} + \frac{1}{16} - \dots \right) = \frac{4}{1 + \frac{1}{2}} = \frac{8}{3}$$

## Power Series:

A power series about  $x = 0$  is a series of the form

$$\sum_{n=0}^{\infty} c_n x^n = c_0 + c_1 x^1 + c_2 x^2 + \dots + c_n x^n + \dots$$

A power series about  $x = a$  is a series of the form

$$\sum_{n=0}^{\infty} c_n(x-a)^n = c_0 + c_1(x-a) + c_2(x-a)^2 + \cdots + c_n(x-a)^n + \cdots$$

in which the center  $a$  and the coefficients  $c_1, c_2, \dots, c_n$  are constants.

**Example:** The series  $\sum_{n=0}^{\infty} x^n$  is a geometric series with first term 1 and ratio  $x$ . It converges to

$$\frac{1}{1+x} = 1 + x + x^2 + x^3 + \cdots + x^n + \cdots \quad \text{for } |x| < 1$$

## Taylor Series & Maclaurin Series:

Let  $f$  be a function with derivatives of all orders throughout some interval

Containing as an interior point. Then the **Taylor Series** generated by  $f$  at  $x = a$  is:

$$\sum_{k=0}^{\infty} f^k(a) \frac{(x-a)^k}{k!} = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^n(a)(x-a)^n}{n!} + \cdots$$

**The Maclaurin Series generated by  $f$  is:**

$$\sum_{k=0}^{\infty} f^k(0) \frac{(x-0)^k}{k!} = f(0) + f'(0)x + \frac{f''(0)x^2}{2!} + \cdots + \frac{f^n(0)x^n}{n!} + \cdots$$

Which is a Taylor series generated by  $a = 0$

**Example:** Find the Taylor series for the following function  $f(x) = \frac{1}{x}$ , at  $a = 2$

Sol:

$$f(x) = \sum_{k=0}^{\infty} f^k(a) \frac{(x-a)^k}{k!}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \cdots + \frac{f^n(a)(x-a)^n}{n!} + \cdots$$

$$f(x) = f(a) = f(2) = \frac{1}{2}$$

$$f'(x) = -\frac{1}{x^2} \rightarrow f'(a) = f'(2) = -\frac{1}{2^2} = -\frac{1}{4}$$

$$f^2(x) = -1(-2x^{-3}) = \frac{2}{x^3} \rightarrow f^2(a) = f^2(2) = \frac{2}{2^3} = \frac{2}{8} = \frac{1}{4}$$

$$f^3(x) = 2(-3x^{-4}) = \frac{-6}{x^4} \rightarrow f^3(a) = f^3(2) = \frac{-6}{2^4} = \frac{-6}{16}$$

Taylor series is:

$$f(x) = \frac{1}{x} = \frac{1}{2} - \frac{1}{4}(x-2) + \frac{2}{8} \frac{(x-2)^2}{2!} - \frac{6}{16} \frac{(x-2)^3}{3!} + \cdots$$

**Example:** Find the Taylor series for the following function

$$f(x) = x^2, \text{ at } a = 4$$

Sol:

$$f(x) = \sum_{k=0}^{\infty} f^k(a) \frac{(x-a)^k}{k!}$$

$$f(x) = f(a) + \dot{f}(a)(x-a) + \frac{\ddot{f}(a)(x-a)^2}{2!} + \dots + \frac{f^n(a)(x-a)^n}{n!} + \dots$$

$$f(x) = f(a) = (4)^2 = 16$$

$$\dot{f}(a) = 2x \rightarrow \dot{f}(a) = 2(4) = 8$$

$$f^2(x) = 2 \rightarrow f^2(4) = 2$$

$$f^3(x) = 0$$

$$f(x) = 16 + 8(x-4) - 2 \frac{(x-4)^2}{2!} + 0$$

**Example** Find Maclaurin series of  $f(x) = x^4 + 5x$

Sol:

For Maclaurin series  $a = 0$

$$\sum_{k=0}^{\infty} f^k(0) \frac{(x-0)^k}{k!} = f(0) + \dot{f}(0)x + \frac{\ddot{f}(0)x^2}{2!} + \dots + \frac{f^n(0)x^n}{n!} + \dots$$

$$f(x) = x^4 + 5x \rightarrow f(0) = 0$$

$$\dot{f}(x) = 4x^3 + 5 \rightarrow \dot{f}(0) = 5$$

$$f^2(x) = 12x^2 \rightarrow f^2(0) = 0$$

$$f^3(x) = 24x \rightarrow f^3(0) = 0$$

Maclaurin series is

$$x^4 + 5x = 0 + 5x + (0) \frac{x^2}{2!} + (0) \frac{x^3}{3!} + \dots = 5x$$

**Example:** Find Taylor series of  $f(x) = \ln x$  at  $a = 1$

Sol:

$$f(x) = \sum_{k=0}^{\infty} f^k(a) \frac{(x-a)^k}{k!}$$

$$f(x) = f(a) + f'(a)(x-a) + \frac{f''(a)(x-a)^2}{2!} + \dots + \frac{f^n(a)(x-a)^n}{n!} + \dots$$

$$f(x) = \ln(x) \rightarrow f(0) = \ln(1) = 0$$

$$f'(x) = \frac{1}{x} \rightarrow f'(1) = \frac{1}{1} = 1$$

$$f^2(x) = -\frac{1}{x^2} \rightarrow f^2(1) = -\frac{1}{1^2} = -1$$

$$f^3(x) = \frac{2}{x^3} \rightarrow f^3(1) = \frac{2}{1^3} = 2$$

Taylor series for  $\ln x$  is:

$$\begin{aligned} \ln x &= x - 1 + \frac{(x-1)^2}{2!}(-1) + \frac{(x-1)^3}{3!}(2) + \dots \\ &= x - 1 - \frac{(x-1)^2}{2!} + \frac{(x-1)^3}{3!} + \dots \end{aligned}$$

## Homework

**-Find Taylor or Maclaurin for the following functions:**

- 1)  $f(x) = \sin 3x$
- 2)  $f(x) = \cos x \text{ at } a = 0$
- 3)  $f(x) = e^x \text{ at } a = 2$
- 4)  $f(x) = x^3 - 2x + 4 \text{ at } a = 1$