Al-Safwa University College

ENGINEERING ANALYSIS

Third Stage

DEPATTMENT OF

COMPUTER TECHNIQUES ENGINEERING

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Chapter One

Laplace Transform

Definition: The Laplace transform of the function $f(t)$ is given by:

$$F(s) = \mathcal{L}[f(t)] = \int_{0}^{\infty} f(t)e^{-st} \, dt$$

Where s is a complex number, $s = \sigma + j \, w$

Laplace Transform of Some Important Function

1) Unit step function u(t):

$$u(t) = \begin{cases} 1 & , t > 0 \\ 0 & , t < 0 \end{cases}$$

$$\mathcal{L}[f(t)] = \int_{0}^{\infty} f(t)e^{-st} \, dt = \int_{0}^{\infty} (1)\, e^{-st} \, dt = \frac{e^{-st}}{-s}\Big|_{0}^{\infty}$$

$$\mathcal{L}[f(t)] = \frac{e^{-s(\infty)}}{-s} - \frac{e^{-s(0)}}{-s} = 0 + \frac{1}{s} = \frac{1}{s}$$

2) Exponential function e^{at} :

$$\mathcal{L}[f(t)] = \mathcal{L}[e^{at}] = \int_0^\infty e^{at} * e^{-st} \ dt = \int_0^\infty e^{at-st} \ dt = \int_0^\infty e^{-t(s-a)} \ dt =$$

$$\frac{e^{-t(s-a)}}{-(s-a)}\Big|_0^\infty = \frac{e^{-\infty(s-a)}}{-(s-a)} - \frac{e^{-0(s-a)}}{-(s-a)} = 0 + \frac{1}{s-a} = \frac{1}{s-a}$$

3) The function sin(at):

$$\mathcal{L}[f(t)] = \mathcal{L}[\sin(at)] = \int_0^\infty \sin(at)\, e^{-st} \ dt = \int_0^\infty (\frac{e^{iat}-e^{-iat}}{2i})e^{-st} \ dt$$

$$= \frac{1}{2i}[\int_0^\infty e^{iat}\, e^{-st} \ dt - \int_0^\infty e^{-iat}\, e^{-st} \ dt]$$

$$= \frac{1}{2i}[\int_0^\infty e^{-t(s-ia)} \ dt - \int_0^\infty e^{-t(s+ia)} \ dt]$$

$$= \frac{1}{2i}\left[\frac{e^{-t(s-ia)}}{-(s-ia)}\Big|_0^\infty - \frac{e^{-t(s+ia)}}{-(s+ia)}\Big|_0^\infty\right]$$

$$= \frac{1}{2i}\left[\frac{e^{-\infty(s-ia)}}{-(s-ia)} - \frac{e^0}{-(s-ia)} + \frac{e^{-\infty(s+ia)}}{s+ia} - \frac{e^0}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\left(0 + \frac{1}{s-ia}\right) + \left(0 - \frac{1}{s+ia}\right)\right] = \frac{1}{2i}\left[\frac{1}{s-ia} - \frac{1}{s+ia}\right]$$

$$= \frac{1}{2i}\left[\frac{(s+ia)-(s-ia)}{(s+ia)(s-ia)}\right] = \frac{1}{2i}\left[\frac{s-s+ia+ia}{s^2+ias-ias-i^2a^2}\right]$$

$$\mathcal{L}[\sin(at)] = \frac{1}{2i}\frac{2ia}{s^2+a^2} = \frac{a}{s^2+a^2}$$

NOTE $\sin at = \dfrac{e^{iat}-e^{-iat}}{2i}$

4) sin(at) and cos(at) function (e^{iat}):

$$e^{iat} = \cos(at) + i\sin(at)$$

$$\mathcal{L}[f(t)] = \mathcal{L}\left[e^{iat}\right] = \int_{0}^{\infty} e^{iat} * e^{-st}\ dt = \int_{0}^{\infty} e^{-t(s-ia)}\ dt = \frac{e^{-t(s-ia)}}{-(s-ia)}\Bigg|_{0}^{\infty}$$

$$= \frac{e^{-\infty(s-ia)}}{-(s-ia)} - \frac{e^{-0(s-ia)}}{-(s-ia)} = 0 + \frac{1}{s-ia} = \frac{1}{s-ia}$$

$$= \frac{1}{s-ia} * \frac{s+ia}{s+ia} = \frac{s+ia}{s^2+a^2} = \frac{s}{s^2+a^2} + i\frac{a}{s^2+a^2} = \mathcal{L}(cosat) + i\mathcal{L}(sinat)$$

5) the function t:

$$\mathcal{L}[f(t)] = \mathcal{L}\,[t] = \int_{0}^{\infty} t * e^{-st}\ dt = \frac{-t\,e^{-ts}}{s} - \frac{e^{-ts}}{s^2}\Bigg|_{0}^{\infty}$$

$$\mathcal{L}[f(t)] \quad = \left(\frac{\infty\,e^{-\infty}}{s} + 0\right) = -\left(0 - \frac{e^0}{s^2}\right) = \frac{1}{s^2}$$

The Laplace transforms for some functions is given in the following table:

$f(t)$	$\mathcal{L}[f(t)]$	$f(t)$	$\mathcal{L}[f(t)]$
1 unit step	$\dfrac{1}{s}$	$\cos(at)$	$\dfrac{s}{s^2 + a^2}$
t	$\dfrac{1}{s^2}$	$\sin(at)$	$\dfrac{a}{s^2 + a^2}$
t^2	$\dfrac{2!}{s^3}$	$\sinh(at)$	$\dfrac{a}{s^2 - a^2}$
t^n	$\dfrac{n!}{s^{n+1}}$	$\cosh(at)$	$\dfrac{s}{s^2 - a^2}$
e^{at}	$\dfrac{1}{s - a}$	$\tanh(at)$	$\dfrac{s^2 + a^2}{(s^2 + a^2)^2}$
$t\, e^{-at}$	$\dfrac{1}{(s + a)^2}$	$(1 - at)e^{-at}$	$\dfrac{s}{(s + a)^2}$

Some Important Properties of Laplace Transform

1) Linearity property:

If c_1 and c_2 any constants while $f_1(t)\, and\, f_2(t)$ are functions with Laplace transformation then:

$$\mathcal{L}[c_1 f_1(t) + c_2 f_2(t)] = c_1\, \mathcal{L}\,[f_1(t)] + c_2\, \mathcal{L}\,[f_2(t)]$$

EXAMPLE: Find Laplace transform of $4t^2 - 3\cos 2t + 5e^{-t}$

Sol:

$$\mathcal{L}\,[4t^2 - 3\cos 2t + 5e^{-t}] = 4\,\mathcal{L}(t^2) - 3\mathcal{L}\,(\cos 2t) + 5\,\mathcal{L}(e^{-t})$$

$$= 4\frac{2!}{s^3} - 3\frac{s}{s^2+4} + 5\frac{1}{s+1} = \frac{8}{s^3} - \frac{3\,s}{s^2+4} + \frac{5}{s+1}$$

EXAMPLE: Find Laplace transform of $4\,e^{5t} + 6\,t^3 - 3\sin 4t + 2\cos 2t$

Sol:

$$\mathcal{L}\,[4\,e^{5t} + 6\,t^3 - 3\sin 4t + 2\cos 2t]$$

$$= 4\,\mathcal{L}\,(e^{5t}) + 6\,\mathcal{L}(t^3) - 3\mathcal{L}(\sin 4t) + 2\,\mathcal{L}(\cos 2t)$$

$$= \frac{4}{s-5} + 6\frac{3!}{s^4} - 3\frac{4}{s^2+16} + \frac{2s}{s^2+4} = \frac{4}{s-5} + \frac{36}{s^4} - \frac{12}{s^2+16} + \frac{2s}{s^2+4}$$

2) Multiplication by e^{at} ($\boldsymbol{Shifting}$):

If $\mathcal{L}\,[f(t)] = F(s)$ then:

$$\mathcal{L}\,[e^{at}\,f(t)] = F(s-a) = F(s)|_{s \to s-a}$$

We obtain $F(s-a)$ by replacing s with $(s-a)$ in the integral.

EXAMPLE: Find: (1) $\mathcal{L}\,(e^{-2t}\sin 3t)$

$\qquad\qquad\qquad\qquad$ (2) $\mathcal{L}\,(e^{4t}\cos 3t)$

$\qquad\qquad\qquad\qquad$ (3) $\mathcal{L}\,(e^{3t}t^2)$

Sol:

(1) $\mathcal{L}\,(e^{-2t}\sin 3t)$

$\qquad$ First we find $\mathcal{L}\,(\sin 3t) = \dfrac{3}{s^2+9}$

$$\mathcal{L}\left(e^{-2t}\sin 3t\right) = \left.\frac{3}{s^2+9}\right|_{s \to s+2} = \frac{3}{(s+2)^2+9} = \frac{3}{s^2+4s+4+9}$$

$$= \frac{3}{s^2+4s+13}$$

(2) $\mathcal{L}\left(e^{4t}\cos 3t\right)$

First we find $\mathcal{L}\left(\cos 3t\right) = \frac{s}{s^2+9}$

$$\mathcal{L}\left(e^{4t}\cos 3t\right) = \left.\frac{s}{s^2+9}\right|_{s \to s-4} = \frac{s-4}{(s-4)^2+9} = \frac{s-4}{s^2-8s+16+9} = \frac{s-4}{s^2-8s+25}$$

(3) $\mathcal{L}\left(e^{3t}t^2\right)$

First we find $\mathcal{L}\left(t^2\right) = \frac{2!}{s^3}$

$$\mathcal{L}\left(e^{3t}t^2\right) = \left.\frac{2!}{s^3}\right|_{s \to s-3} = \frac{2}{(s-3)^3}$$

3) Multiplication by t^n:

$$\mathcal{L}\left[t^n f(t)\right] = (-1)^n \frac{d^n}{ds^n} F(s)$$

EXAMPLE: Find $\mathcal{L}\left(t^2 \cos at\right)$

Sol:

$$n = 2, \ f(t) = \cos at \ \to F(s) = \frac{s}{s^2+a^2}$$

$$\mathcal{L}(t^2 \cos at) = (-1)^2 \frac{d^2}{ds^2}\left(\frac{s}{s^2+a^2}\right)$$

$$\frac{d}{ds}\left(\frac{s}{s^2+a^2}\right) = \frac{(s^2+a^2)*1-s*2s}{(s^2+a^2)^2} = \frac{s^2+a^2-2s^2}{(s^2+a^2)^2} = \frac{a^2-s^2}{(s^2+a^2)^2}$$

$$\frac{d^2}{ds^2}\left[\frac{a^2-s^2}{(s^2+a^2)^2}\right] = \frac{[(s^2+a^2)^2*-2s]-[(a^2-s^2)*2(s^2+a^2)*2s]}{(s^2+a^2)^4}$$

$$=\frac{[-2s(s^2+a^2)^2]-4s[(a^2-s^2)(s^2+a^2)]}{(s^2+a^2)^4} = \frac{-2s\ (s^2+a^2)[(s^2+a^2)+2(a^2-s^2)]}{(s^2+a^2)^4}$$

$$=\frac{-2s[\ s^2+a^2+2a^2-2\ s^2]}{(s^2+a^2)^3} = \frac{-2s\ (3a^2-s^2)}{(s^2+a^2)^3}$$

4) **Initial Value theorm**:

When $t=0$, $f(0) = \lim_{t\to0} f(t) = \lim_{s\to\infty} s\ F(s)$

EXAMPLE: Find the initial value of F(s) if $F(s) = \dfrac{1}{s(s^2+1)}$

  Sol:

$$f(0) = \lim_{t\to0} f(t) = \lim_{s\to\infty} s\ F(s) = \lim_{s\to\infty} s*\frac{1}{s(s^2+1)}$$

$$= \frac{1}{s^2+1} = \frac{1}{\infty^2+1} = \frac{1}{\infty} = 0$$

5) **Final value theorm (steady state):**

    When $t=\infty$, $\lim_{t\to\infty} f(t) = \lim_{s\to0} s\ F(s)$

EXAMPLE: Find the steady state value of F(s) if $F(s) = \dfrac{1}{s(s^4+2s+7)}$

  Sol:

$$\lim_{s\to0} s\ F(s) = \lim_{s\to0} s\ \frac{1}{s(s^4+2s+7)} = \frac{1}{0+2*0+7} = \frac{1}{7}$$

6) **Laplace transform for Derivative:**

$$\mathcal{L}[\acute{f}(t)] = SF(s) - f(0)$$

$$\mathcal{L}[f^2(t)] = S^2 F(s) - Sf(0) - \acute{f}(0)$$

$$\mathcal{L}[f^3(t)] = S^3 F(s) - S^2 f(0) - S \acute{f}(0) - f^2(0)$$

Therefore, in general:

$$\mathcal{L}[f^n(t)] = S^n F(s) - S^{n-1} f(0) - S^{n-2} \acute{f}(0) - \cdots - S f^{n-2}(0) - f^{n-1}(0)$$

EXAMPLE: Find $\mathcal{L}[\acute{f}(t)]$ if $f(t) = -\cos t$

Sol:

$$\mathcal{L}[\acute{f}(t)] = SF(s) - F(0) = S[\mathcal{L}f(t)] - f(0) = S[\mathcal{L}(-cost)] - f(0)$$

$$\mathcal{L}[\acute{f}(t)] = -S\frac{s}{s^2+1} - (-\cos 0) = -\frac{s^2}{s^2+1} + \cos 0 = \frac{-s^2}{s^2+1} + 1$$

$$\mathcal{L}[\acute{f}(t)] = \frac{-s^2 + s^2 + 1}{s^2 + 1} = \frac{1}{s^2 + 1}$$

7) Laplace Transform of Integral:

If $F(t) = \mathcal{L}[f(t)]$ then:

$$\mathcal{L}[\int_0^t f(t)dt] = \frac{1}{s} F(s)$$

EXAMPLE: Find $\mathcal{L}\left[\int_0^t (\sin t)\, dt \right]$

Sol: $\mathcal{L}\left[\int_0^t (\sin 3\, t)\, dt \right] = \frac{F(s)}{s} = \frac{1}{s}\left(\frac{3}{s^2+9}\right) = \frac{3}{s(s^2+9)}$

Inverse of Laplace Transform

If the Laplace transform of a function f (t) is F(s) then f (t) is called the inverse of Laplace transform of f (s) and we write symbolically:

$$f(t) = \mathcal{L}^{-1}[F(s)]$$

where $\mathcal{L}^{-1}$ is called the inverse of Laplace transformation operator. f (t) can be found by using properties or partial fraction method.

EXAMPL: Find the inverse of Laplace transform of F(s) if $F(s) = \frac{s+1}{(s+2)^2}$

Sol:

$$F(s) = \frac{s+1}{(s+2)^2} = \frac{1}{(s+2)^2} + \frac{s}{(s+2)^2}$$

$$f(t) = \mathcal{L}^{-1}\left[\frac{1}{(s+2)^2}\right] + \mathcal{L}^{-1}\left[\frac{s}{(s+2)^2}\right] = (1-2t)e^{-2t} + t\,e^{-2t}$$

$$f(t) = e^{-2t} - 2te^{-2t} + +t\,e^{-2t} = e^{-2t}(1-t)$$

Rule of Partial Fraction

1- $s + a = \frac{A}{s+a}$

2- $(s + a)^2 = \frac{A}{s+a} + \frac{B}{(s+a)^2}$

3- $(s + a)^3 = \frac{A}{s+a} + \frac{B}{(s+a)^2} + \frac{c}{(s+a)^3}$

4- $(s^2 + ps + q) = \frac{As+B}{s^2+ps+q}$

5- $(s^2 + ps + q)^2 = \frac{As+B}{s^2+ps+q} + \frac{Cs+D}{(s^2+ps+q)^2}$

EXAMPL: Find $\mathcal{L}^{-1}\left[\frac{2s^2+11s+5}{s(s+2)(s+3)}\right]$

Sol:

$$\mathcal{L}^{-1}\left[\frac{2s^2+11s+5}{s(s+2)(s+3)}\right] = \frac{A}{s} + \frac{B}{s+2} + \frac{c}{s+3}$$

$$A = \lim_{s\to0} \frac{2s^2+11s+5}{(s+2)(s+3)} = \frac{0+0+5}{(0+2)(0+3)} = \frac{5}{6}$$

$$B = \lim_{s\to-2} \frac{2s^2+11s+5}{s(s+3)} = \frac{2(-2)^2+11(-2)+5}{-2(-2+3)} = \frac{-9}{-2} = \frac{9}{2}$$

$$C = \lim_{s\to-3} \frac{2s^2+11s+5}{s(s+2)} = \frac{2(-3)^2+11(-3)+5}{-3(-3+2)} = \frac{-10}{3}$$

$$\mathcal{L}^{-1}\left[\frac{A}{s} + \frac{B}{s+2} + \frac{c}{s+3}\right] = \frac{5}{6} * \frac{1}{s} + \frac{9}{2} * \frac{1}{s+2} - \frac{10}{3} * \frac{1}{s+3}$$

$$= \frac{5}{6}(1) + \frac{9}{2}e^{-2t} - \frac{10}{3}e^{-3t}$$

EXAMPLE: Find $\mathcal{L}^{-1}\left[\frac{s^2+s+1}{s^2-5s+6}\right]$

Sol:

$$\mathcal{L}^{-1}\left[\frac{s^2+s+1}{s^2-5s+6}\right] = \mathcal{L}^{-1}\left[\frac{s^2+s+1}{(s-2)(s-3)}\right] = \frac{A}{s-2} + \frac{B}{s-3}$$

$$A = \lim_{s\to2} \frac{s^2+s+1}{(s-3)} = \frac{4+2+1}{2-3} = -7$$

$$B = \lim_{s \to 3} \frac{s^2+s+1}{(s-2)} = \frac{9+3+1}{3-2} = 13$$

$$\mathcal{L}^{-1}\left[\frac{s^2+s+1}{(s-2)(s-3)}\right] = \mathcal{L}^{-1}\left[\frac{A}{s-2}\right] + \mathcal{L}^{-1}\left[\frac{B}{s-3}\right]$$

$$= \frac{-7}{s-2} + \frac{13}{s-3} = -7\,e^{2t} + 13\,e^{3t}$$

Solution of Differential Equation Using Laplace Transform

Here, we use the derivative property as follows:

$$\mathcal{L}[\acute{f}(t)] = SF(s) - f(0)$$

$$\mathcal{L}[f^2(t)] = S^2F(s) - Sf(0) - \acute{f}(0)$$

$$\mathcal{L}[f^3(t)] = S^3F(s) - S^2f(0) - S\,\acute{f}(0) - f^2(0)$$

EXAMPLE: Solve the following differential equation using Laplace

transform: $\acute{y} + 2\acute{y} + y = t \ \ with \ y(0) = 0 \ , \acute{y}(0) = 1$

Sol:

Taking the Laplace transform of the two sides, we get

$$[s^2Y(s) - sy(0) - \acute{y}(0)] + 2[sY(s) - y(0)] + Y(s) = \frac{1}{s^2}$$

$$s^2Y(s) - s(0) - 1 + 2sY(s) - 2(0) + Y(s) = \frac{1}{s^2}$$

$$s^2 Y(s) + 2sY(s) + Y(s) = \frac{1}{s^2} + 1$$

$$Y(s)[s^2 + 2s + 1] = \frac{1 + s^2}{s^2} \rightarrow Y(s) = \frac{1 + s^2}{s^2(s^2 + 2s + 1)}$$

$$Y(s) = \frac{1 + s^2}{s^2(s^2 + 2s + 1)} = \frac{1 + s^2}{s^2(s + 1)(s + 1)} = \frac{1 + s^2}{s^2(s + 1)^2}$$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2}$$

$$Y(s) = \frac{As(s+1)^2 + B(s+1)^2 + Cs^2(s+1) + Ds^2}{s^2(s+1)^2}$$

$$Y(s) = \frac{As^3 + 2As + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Ds^2}{s^2(s+1)^2}$$

$$s^2 + 1 = As^3 + 2As + As + Bs^2 + 2Bs + B + Cs^3 + Cs^2 + Ds^2$$

$$s^2 + 1 = s^3(A + C) + s^2(2A + B + C + D) + s(A + 2B) + B$$

$$s^3(A + C) = 0 \rightarrow A + C = 0 \rightarrow A = -C \ \dots (1)$$

$$s^2(2A + B + C + D) = s^2 \rightarrow 2A + B + C + D = 1 \ \dots (2)$$

$$s(A + 2B) = 0 \rightarrow A + 2B = 0 \dots (3)$$

$$B = 1 \dots (4) \ , we \ put \ (1) in \ (3) \ and \ get \ A = -2(1) = -2$$

Then we get the value of C by eq.(1), $C = -A = 2$

Now the value of D can be obtained by requiting A, B, C in eq.(2) to get:

$$2A + B + C + D = 1 \rightarrow 2(-2) + 1 + 2 + D = 1 \rightarrow D = 2$$

$$Y(s) = \frac{A}{s} + \frac{B}{s^2} + \frac{C}{s+1} + \frac{D}{(s+1)^2} = Y(s) = \frac{-2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{2}{(s+1)^2}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{-2}{s} + \frac{1}{s^2} + \frac{2}{s+1} + \frac{2}{(s+1)^2}\right] = -2 + t + 2e^{-t} + 2te^{-t}$$

EXAMPLE: Solve the following differential equation using Laplace transform

$$\acute{y} + y = 4 \,, with \; y(0) = 0$$

Sol:

Take Laplace transform for both sides

$$sY(s) + y(0) + Y(s) = \frac{4}{s}$$

$$sY(s) + Y(s) = \frac{4}{s} \;\rightarrow Y(S)(s+1) = \frac{4}{s}$$

$$Y(S) = \frac{4}{s(s+1)} = \frac{A}{s} + \frac{B}{s+1} = \frac{A(s+1) + Bs}{s(s+1)}$$

$$Y(s) = \frac{As + A + Bs}{s(s+1)} = \frac{s(A+B) + A}{s(s+1)}$$

$$s(A+B) + A = 4$$

$$s(A+B) = 0 \rightarrow A + B = 0 \; \rightarrow A = -B \; ... \,(1)$$

$$A = 4 \;\; \rightarrow B = -4 \; ... \,(2)$$

$$Y(S) = \frac{A}{s} + \frac{B}{s+1} = \frac{4}{s} - \frac{4}{s+1}$$

$$y(t) = \mathcal{L}^{-1}\left[\frac{4}{s} - \frac{4}{s+1}\right] = 4 - 4e^{-t}$$

HOMEWORK

1- Find $\mathcal{L}(e^{-t}\sinh at)$

2- If $f(t) = \sin 2t$ find $\mathcal{L}[f^2(t)]$

3- find $\mathcal{L}[f(t)]$ $\quad if \quad f(t) = e^{-t}(\sin 2t - \cos 4t))$

4- If $\quad F(s) = \dfrac{1}{s^2+5s-6}$ find inverse of Laplace transform.

5- Solve the following differential equations:

a) $\acute{y} + 2y = 5$, $y(0) = 0$, $\acute{y}(0) = 0$

b) $\acute{y} + 3y = 4e^t$, $y(0) = 0$

Chapter Two

Z- Transform

Z- Transform: Is a representation of a discrete signal and play an important role in the analysis of discrete time signal.

Definition: Z- Transform of discrete time signal X (n) is defined as:

$$x(z) = \sum_{n=-\infty}^{\infty} x(n)\ z^{-n}$$

Where **z** is complex variable: $z = x + iy \quad or \quad z = re^{iw}$

From the above definition of Z.T. it is clear that ZT is power series & it exist for only for those values of z for which X(z) attains finite value (convergence) ,which is defined by Region of convergence (ROC).

Region of Convergence (ROC)

Region of Convergence is set of those values of z for which power series x (z) converges. OR for which power series, x (z) attains finite value.

The poles of a z- transform are the value of **z** for which if $x(z) = \infty$ and the zeros of z- transform are the values of **z** for which if $x(z) = 0$

Z- Transform of Some Functions

1- $Z[\delta(n)] = \sum_{n=0}^{\infty} \delta(n) z^{-1} = 1$

2- $Z[u(n)] = \sum_{n=0}^{\infty} 1 * z^{-n} = (z^{-1})^n$

$= 1 + z + z^{-1} + z^{-2} + \cdots = \dfrac{1}{1-z^{-1}} = \dfrac{z}{z-1}$, $ROC: \quad |z| > 1$

3- $Z[e^{-an}] = \dfrac{1}{1-z^{-1}e^{-a}} = \dfrac{z}{z-e^{-a}}$, $ROC: \quad |z| > e^{-a}$

4- $Z[a^n] = \sum_{n=0}^{\infty} a^n z^{-n} = \dfrac{1}{1-az^{-1}} = \dfrac{z}{z-a}$, $ROC: |Z| > a$

5- $Z[n] = \sum_{n=0}^{\infty} n z^{-n} = \dfrac{z^{-1}}{(1-z^{-1})^2} = \dfrac{z}{(z-1)^2}$, $ROC: |Z| > 1$

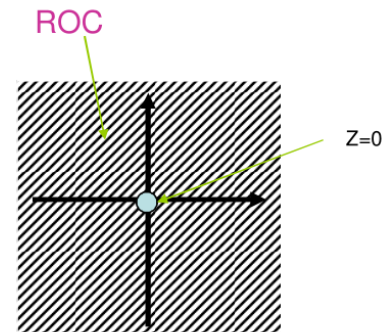**EXAMPLE:** If $x(n) = \{1, 2, 5, 7, 0, 1\}$, find Z transform.

Sol:

$x(z) = \sum_{n=-\infty}^{\infty} x(n) \; z^{-n}$

$x(z) = x(0)z^0 + x(1)z^{-1} + x(2)z^{-2} + x(3) z^{-3} + x(4)z^{-4} + x(4)z^{-5}$

$x(z) = 1 + 2z^{-1} + 5z^{-2} + 7 z^{-3} + z^{-5}$

$= 1 + \dfrac{2}{z} + \dfrac{5}{z^2} + \dfrac{7}{z^3} + \dfrac{1}{z^5}$

ROC: $Z \neq 0$

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-In this case $x(z)$ is finite for all values of z, except

$|z| = 0$. Because at z = 0, $x(z)= \infty$.
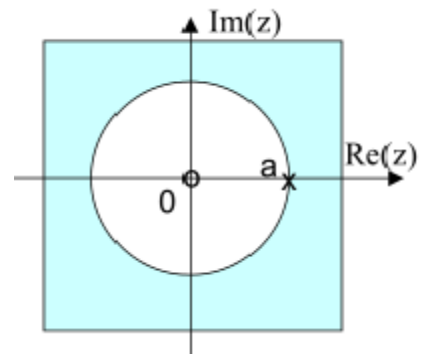
Thus ROC is entire z-plane except $|z| = 0$.

EXAMPLE: Find Z transform of x(n) and ROC if :

$$x(n) = \begin{cases} a^n , & n \geq 0 \\ 0, & n < 0 \end{cases}$$

Sol:

$$x(z) = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (a\, z^{-1})^n = \frac{z}{z-a}$$
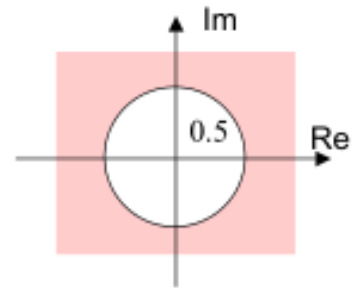
$ROC: |z| > a$

EXAMPLE: Find Z transform of x (n) and ROC if : $x(n) = \left(\frac{1}{2}\right)^n u(n)$

Sol:

$u(n) = 1$

$$x(z) = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n} = \frac{z}{z-\frac{1}{2}}$$

$ROC: |z| > \frac{1}{2}$

EXAMPLE: Find Z transform of x (n) and ROC if:

$$x(n) = -a^n\, u(-n - 1) = \begin{cases} 0 & , n \geq 0 \\ -a^n, & n \leq -1 \end{cases}$$
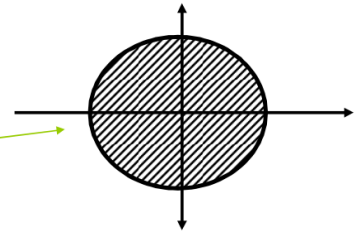
Sol:

$$x(z) = \sum_{n=-\infty}^{-1}(-a^n)z^{-n} = -\sum_{n=1}^{\infty}(a^{-n}z^n)$$

$$x(z) = -\left(\sum_{n=0}^{\infty}a^{-n}z^n - 1\right) = -\left(\frac{a}{a-z} - 1\right)$$

$$= -\frac{a-a+z}{a-z} = -\frac{z}{a-z} = \frac{z}{z-a}$$

$$ROC: \; |z| < |a|$$

ROC is inside the
circle |z|=|a|

EXAMPLE: Find Z-Transform of $x\,(n)$
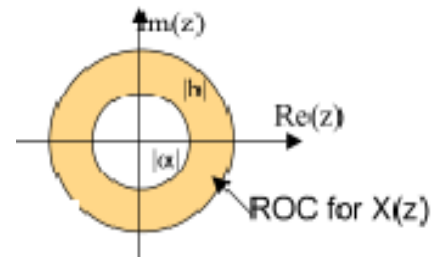
and ROC if: $x(n) = a^n\, u(n) + b^n\, u(-n-1)$

Sol:

$$x(z) = \sum_{n=0}^{\infty}a^n z^{-n} + \sum_{n=-\infty}^{-1}b^n\ z^{-n} = \sum_{n=0}^{\infty}a^n z^{-n} + \sum_{n=1}^{\infty}b^{-n}z^n$$

$$= \sum_{n=0}^{\infty}a^n z^{-n} + \sum_{n=0}^{\infty}b^{-n}\ z^n - 1 = \frac{z}{z-a} + \frac{b}{b-z} - 1$$

$$x(z) = \frac{z}{z-a} + \frac{b-b+z}{b-z} = \frac{z}{z-a} + \frac{z}{b-z}$$

$$ROC: \; |z| > a\,, \;\; |z| < b \;\; \text{or} \;\; a < |z| < b$$

Im(z)

|h|

Re(z)

|α|

ROC for X(z)

Properties of Z-transform

1- Linearity: $Z\{a_1 x_1(n) \pm a_2 x_2(n)\} = a_1 x_1(z) \pm a_2 x_2(z)$

EXAMPLE: $Z\left[\frac{1}{a}(1 - e^{-an})\right] = \frac{1}{a}\left[\frac{z}{z-1} - \frac{z}{z-e^{-a}}\right]$

2- Time Shifting: $Z[x(n-m)] = Z^{-m}x(z)$

 EXAMPLE: $\qquad\qquad Z[x(n-1)] = Z^{-1}x(z)$

3- Multiplication by exponential : $Z[x(n)e^{\pm a_n}] = x(Z e^{\mp a})$

EXAMPLE: $Z[n\ e^{a_n}] = \dfrac{z\ e^{-a}}{(ze^{-a}-1)^2} = \dfrac{z\ e^a}{(z-e^a)^2}$

4- Multiplication by n: $Z[n\ x(n)] = -Z\dfrac{d}{dz}x(z)$

EXAMPLE: $Z[n\ e^{a_n}] = -Z\dfrac{d}{dz}\left(\dfrac{z}{z-e^a}\right) = -z\left(\dfrac{z-e^a*1-z*1}{(z-e^a)^2}\right) = \dfrac{z\ e^a}{(z-e^a)^2}$

5- Time reversal: $x(-n) = x(z^{-1})\ \ and\ x(n) = x(z)$

 EXAMPLE: $x(n) = u(-n)\ \ and\ u(-n) = \dfrac{1}{1-z}\ ,\quad |z| < 1$

 EXAMPLE: $x(n) = u(n)\ \ and\ u(n) = \dfrac{1}{1-z^{-1}}\ ,\qquad |z| > 1$

6- Convolution: Z-Transform Convolution Theorem

$$y[n] = x(z) * h(z) \overset{z}{\Leftrightarrow} X(z)H(z) = Y(z)$$

Or: $\qquad\qquad x_1(z) * x_2(z) \overset{z}{\Leftrightarrow} X_1(z)X_2(z)$

EXAMPLE: Find x(n) using convolution theorem if $X(z) = \frac{z}{(z-1)^2}$

Sol:

$$X(z) = \frac{z}{(z-1)^2} = \frac{z}{z-1} \cdot \frac{1}{z-1} = x_1(z).x_2(z) = u(n).u(n-1)$$

EXAMPLE: Find out convolution of two sequences given below:

$$x(n) = \{2\ 1 - 1\ 0\ 3\}\ , h(n) = \{\ 1\ \ 2\ -1\}$$
$$\qquad\qquad\quad\uparrow\qquad\qquad\qquad\quad\uparrow$$

Sol:

$$X(z) = 2z^0 + z^{-1} - z^{-2} + 0 + 3z^{-4} = 2 + z^{-1} - z^{-2} + 3z^{-4}$$

$$H(z) = z + 2z^0 - z^{-1} = z + 2 - z^{-1}$$

Using convolution property

$$X(z).H(z) = (2 + z^{-1} - z^{-2} + 3z^{-4})(z + 2 - z^{-1}\ )$$

$$X(z).H(z)$$

$$= 2z + 1 - z^{-1} + 3z^{-3} + 4 + 2z^{-1} - 2z^{-2} + 6z^{-4} - 2z^{-1} - z^{-2}$$

$$+ z^{-3} - z^5 = 2z + 5 - z^{-1} + 4z^{-3} - 3z^{-2} - z^5 + 6z^{-4}$$

The Inverse of Z-Transform

Methods of obtaining the inverse of Z-transform are:

1- Partial fraction method.

2- Power fraction method.

3- Discrete convolution.

Partial Fraction Method

$x(n) = Z^{-1}[x(z)]$

$$\frac{x(z)}{z} = \frac{A}{z - \ell_1} + \frac{B}{z - \ell_2}$$

Where ℓ_1 and ℓ_2 are functions.

EXAMPLE: Find the inverse of Z-transform if $x(z) = \frac{2(z^2+z)}{z^2-1.5\,z+0.5}$

Sol:

$$x(z) = \frac{2(z^2+z)}{z^2-1.5\,z+0.5} = \frac{2z(z+1)}{z^2-1.5\,z+0.5}$$

$$\frac{x(z)}{z} = \frac{2(z+1)}{(z-1)(z-0.5)} = \frac{A}{z-1} + \frac{B}{z-0.5}$$

$$A = \lim_{z\to 1} \frac{2(z+1)}{z-0.5} = \frac{2(1+1)}{1-0.5} = \frac{4}{0.5} = 8$$

$$B = \lim_{z\to 0.5} \frac{2(z+1)}{z-1} = \frac{2(0.5+1)}{0.5-1} = \frac{3}{-0.5} = -6$$

$$x(z) = 8\frac{z}{z-1} - 6\frac{z}{z-0.5}$$

$$x(n) = 8\,u(n) - 6\,(0.5)^n$$

EXAMPLE: Find x(n) if $\quad x(z) = \dfrac{z}{3z^2 - 4z + 1}$

  Sol:

$$\frac{x(z)}{z} = \frac{1}{3z^2 - 4z + 1} = \frac{1}{(3z-1)(z-1)} = \frac{A}{3z-1} + \frac{B}{z-1}$$

$$A = \lim_{z \to \frac{1}{3}} \frac{1}{z-1} = \frac{1}{\frac{1}{3}-1} = \frac{1}{-\frac{2}{3}} = -\frac{3}{2}$$

$$B = \lim_{z \to 1} \frac{1}{3z - 1} = \frac{1}{3*1 - 1} = \frac{1}{2}$$

$$\frac{x(z)}{z} = \frac{A}{3z-1} + \frac{B}{z-1} = -\frac{3}{2}\frac{1}{3z-1} + \frac{1}{2}\frac{1}{z-1}$$

$$x(z) = -\frac{3}{2}\frac{z}{3z-1} + \frac{1}{2}\frac{z}{z-1} = -\frac{3}{2}\frac{z}{3\left(z-\frac{1}{3}\right)} + \frac{1}{2}\frac{z}{z-1} = -\frac{1}{2}\frac{z}{z-\frac{1}{3}} + \frac{1}{2}\frac{z}{z-1}$$

$$x(n) = -\frac{1}{2}\left(\frac{1}{3}\right)^n + \frac{1}{2}\,u(n)$$

Stability of System

In general, to test the stability of digital system, just check that all the poles

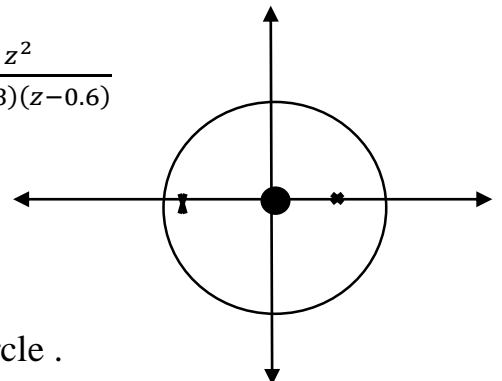within the unit circle. If any are not, then the system is unstable.

For example: if

$$x(z) = \frac{1}{1 + 0.2\,z^{-1} + 0.48\,z^{-2}} * \frac{z^2}{z^2} = \frac{z^2}{z^2 + 0.2\,z + 0.48} = \frac{z^2}{(z+0.8)(z-0.6)}$$

The zeros: $z_1 = 0$

The poles: $p_1 = -0.8, \quad p_2 = 0.$

The system is stable because all poles inside unite circle .

EXAMPLE: Find poles and zeros for $x(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}}$

Sol:

$$x(z) = \frac{z}{z-\frac{1}{2}} + \frac{z}{z+\frac{1}{3}} = \frac{z\left(z+\frac{1}{3}\right)+z\left(z-\frac{1}{2}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} = \frac{z^2+\frac{1}{3}z+z^2-\frac{1}{2}z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

$$x(z) = \frac{2z^2-\frac{1}{6}z}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)} = \frac{z\left(2z-\frac{1}{6}\right)}{\left(z-\frac{1}{2}\right)\left(z+\frac{1}{3}\right)}$$

The zeros:　　$z_1 = 0$, $z_2 = \frac{1}{12}$

The poles:　　$p_1 = \frac{1}{2}$,　　　$p_2 = -\frac{1}{3}$

The system is stable.
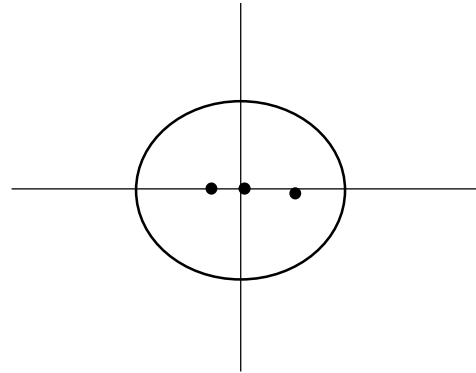
HOMEWORK

1- Find Z-transform if $x(n) = b^{|n|}$

2-If $x(z) = 2z + 5z^{-1} + 6z^{-3} - z^4$ find $x(n)$.

3-Find zeros and poles of $x(z) = \dfrac{1}{\left(1-\frac{1}{3}z^{-1}\right)(1-2z^{-2})}$

$4 -$ Find x(n) if $x(z) = \dfrac{z + 2}{(z + 2)(z - 4)}$

$5 -$ If $x(z) = \dfrac{z^{-2}}{(2-z^{-2})}$ find:

(a) $[x(n) + x(n - 3)] * x(n)$

(b) $x(n + 3) + \delta(n - 1)$

6- If $x(n) = \{2 \quad 1 \quad -1 \quad 0 \quad 3\}, h(n) = \{1 \quad 2 \quad -1\}$,

Find out convolution of two sequences given above.

Chapter Three

Probability and Statistics

Definition: If the sample space S of an experiment consists of finitely many outcomes (points) that are equally likely, then the probability of an event A is:

$$p(A) = \frac{number\ of\ points\ in\ A}{number\ of\ points\ in\ S} = \frac{h}{n}$$

From this definition it follows immediately that, in particular, $p(s) = 1$

EXAMPLE: Consider again the weather example, with S = {rain, snow, clear}. Suppose that the probability of rain is 40%, the probability of snow is 15%, and the probability of a clear day is 45%.

Sol:

For this example, of course P(∅) = 0, i.e., it is impossible that nothing will happen tomorrow. Also P ({rain, snow, clear}) = 1, because we are assuming that exactly one of rain, snow, or clear must occur tomorrow.

$p(rain) = \frac{40}{100} = 0.4$ and $p(snow) = \frac{15}{100} = 0.15, p(clear) = \frac{45}{100} = 0.45$

If the probability of non-occurrence of event is q then:

$$p + q = 1 \rightarrow p = 1 - q$$

$$\therefore p(A) = 1 - q$$

Sample Space

The sample space of the experiment consists of the set of the all possible outcomes.

General Definitions of Probability

Given a sample space S, with each event A of S there is associated a number called the probability of A, such that the following axioms of probability are satisfied.

1) For every A in S, $0 \leq p(A) \leq 1$

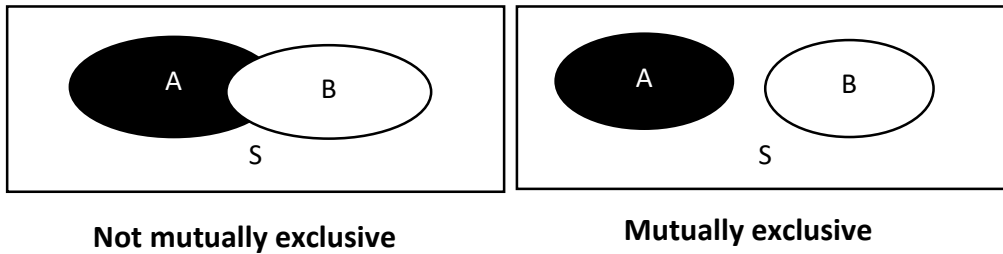**2) The entire sample space S has the probability**

$$p(S) = 1$$

3) For mutually exclusive events A and B

$$p(A \cup B) = p(A) + p(B) \quad \text{where} \quad (A \cap B = \emptyset)$$

4) For not mutually exclusive events A and B in a sample space,

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

Where: $p(A \cap B) = p(A) * p(B)$

Not mutually exclusive   **Mutually exclusive**

Complementation Rule

For an event A and its complement A^c in a sample space S,

$$p(A^c) = 1 - p(A)$$
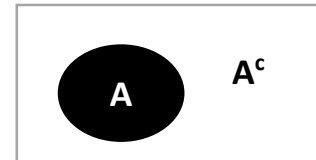
EXAMPLE: If player **A** hits goal with probability (1/4) and player **B** with (2/5) what is the probability of hitting the goal if **A** and **B** are hitting the   goal in the same time?

Sol:

$$p(A \cup B) = p(A) + p(B) - p(A \cap B)$$

$$p(A \cup B) = p(A) + p(B) - p(A) * p(B)$$

$$p(A \cup B) = \frac{1}{4} + \frac{2}{5} - \left(\frac{1}{4} * \frac{2}{5}\right) = \frac{11}{20}$$

Conditional Probability (Independent Events)

Often it is required to find the probability of an event **B** under the condition that an event **A** occurs. This probability is called the conditional probability of **B** given **A** and is denoted by $p(A/B)$**.** Thus:

$$p(A/B) = \frac{p(A \cap B)}{p(A)} \qquad where\ p(A) \neq 0$$

Similarly, the conditional probability of **A** given **B** is:

$$p(B/A) = \frac{p(A \cap B)}{p(B)} \qquad where\ p(B) \neq 0$$

EXAMPLE: Suppose a class contains 60% girls and 40% boys. Suppose that 30% of the girls have long hair, and 20% of the boys have long hair. A student is chosen uniformly at random the class. What is the probability that the chosen student will have long hair**?**

Sol: Let A_1 be the set of girls and A_2 be the set of boys. Then {A_1, A_2} is a partition of the class. We further let B be the set of all students with long hair.

$$P(B) = P(A_1)P(B \mid A_1) + P(A_2)P(B \mid A_2)$$

$$= (0.6)(0.3) + (0.4)(0.2) = 0.26$$

Multiplication Rule

If **A** and **B** are events in a sample space **S** and $p(A) \neq 0, p(B) \neq 0$ then:

$$p(A \cap B) = P(A)P(A/B) = P(B)P(B/A)$$

EXAMPLE: In producing screws, let **A** mean "screw too slim" and **B** "screw too short." Let p(A) = 0.1 and let the conditional probability that a slim screw is also too short be $P(B/A) = 0.2$. What is the probability that a screw that we pick randomly from the lot produced will be both too slim and too short?

Sol:

$$p(A \cap B) = p(A)P(B/A)$$

$$p(A \cap B) = 0.1 * 0.2 = 0.02 = 2\%$$

EXAMPLE: One bag contains 4 white balls and 2 black balls and another bag contain 3 white balls and 5 black balls. If one ball is drown from each bag, find the probability that:

    1- Both are white   2- Both are black.

Sol:

$p(w_1) = \frac{4}{6}$, $p(b_1) = \frac{2}{6}$

$p(w_2) = \frac{3}{8}$, $p(b_2) = \frac{5}{8}$

bag (1)		bag (2)	
w₁	b₁	w₂	b₂
4	2	3	5
6		8	

1) $p(w_1 w_2) = p(w_1)\, p(w_2) = \frac{4}{6} * \frac{3}{8} = \frac{1}{4}$

2) $p(b_1 b_2) = p(b_1)\, p(b_2) = \frac{2}{6} * \frac{5}{8} = \frac{5}{24}$

EXAMPLE: A box contains 10 white ball and 5 red balls, a second box contains 20 white and 20 red balls. We select at random box and pick a ball. What is the probability that this ball is white?

Sol:

$p(box_1) = p(box_2) = \frac{1}{2}$

$p(w) = p(w_1\ box\ _1) + p(\ w_2\ box_2) =$

$p(w) = \frac{10}{15} * \frac{1}{2} + \frac{20}{40} * \frac{1}{2} = \frac{7}{12}$

bag (1)		bag (2)	
w₁	r₁	w₂	r₂
10	5	20	20
15		40	

Proof: $\quad p(A) = 1 - p(A^c) \text{ if } p(S) = 1.$

Sol:

We can draw the sample space into two events A $\;and\;$ A^c , then

$\quad S = A \cup A^c$

$\quad P(S) = P(A \cup A^c) = P(A) + P(A^c)$

$\quad 1 = P(A) + P(A^c) \rightarrow \quad \therefore \; P(A) = 1 - P(A^c)$
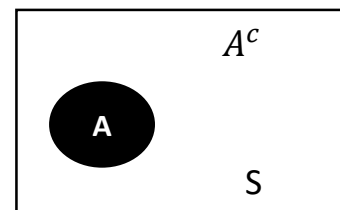
Proof: $\quad P(\emptyset) = 0 \quad \text{if } \emptyset \; and \; A \text{ are two exclusive}$

Sol:

$\quad A \cup \emptyset = A$

$\quad P(A \cup \emptyset) = P(A)$

$\quad P(A) + P(\emptyset) = P(A)$

$\quad P(\emptyset) = P(A) - P(A) = 0$

EXAMPLE: If the probability of the student A to solve a problem is (4/5) and the probability of the student B to solve it is (2/3), what is the probability of this problem if two students are solved together?

Sol:

$$p(A) = \frac{4}{5} \qquad\qquad , p(B) = \frac{2}{3}$$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cup B) = P(A) + P(B) - P(A).P(B)$$

$$P(A \cup B) = \frac{4}{5} + \frac{2}{3} - \left(\frac{4}{5} * \frac{2}{3}\right) = \frac{12 + 10}{15} - \frac{8}{15} = \frac{14}{15}$$

EXAMPLE: If the probability of the student success in math is (2/3) and the probability of the student success in physics is (4/9), and the probability of the student success in any exam is (4/5)

1- What is the probability of success in both exams?
2- What the probability is of failed in math?

Sol:

1- $p(A) = \frac{2}{3} \qquad\qquad , p(B) = \frac{4}{9} \qquad , P(A \cup B) = \frac{4}{5}$

$$P(A \cup B) = P(A) + P(B) - P(A \cap B)$$

$$P(A \cap B) = P(A) + P(B) - P(A \cup B)$$

$$P(A \cap B) = \frac{2}{3} + \frac{4}{9} - \frac{4}{5} = \frac{14}{45}$$

2- $P(A^c) = 1 - P(A) = 1 - \frac{2}{3} = \frac{1}{3}$

Permutations and Combinations

Permutations:

A permutation of given things (elements or objects) is an arrangement of these things in a row in some order.

(a) Different things: The number of permutations of n different things taken all at a time is $n! = 1.2.3 \dots n$

(b) The number of different permutations of **n** different things taken **k** at a time without repetitions is:

$$\substack{n \\ k}p = \frac{n!}{(n-k)!}$$

And with repetitions:

$$\substack{n \\ k}p = n^k$$

EXAMPLE: In how many groups the letters a, b, c can be arranged taken two letters at time?

Sol:

$$\substack{n \\ k}p = \substack{3 \\ 2}p = \frac{3!}{(3-2)!} = 6$$

- **If $n = n_1 + n_2 + n_3 + \cdots + n_r$,**

therefore the number of permutations for this things is equal to:

$$\binom{n}{n_1 \quad n_2 \quad n_3 \dots n_r} = \frac{n!}{n_1! \, n_2! \, n_3! \dots n_r!}$$

EXAMPLE: In how many way can be arranged the letters of **STATISTICS?**

Sol:

$$\binom{n}{n_1 \quad n_2 \quad n_3 \dots n_r} = \frac{10!}{3!3!2!1!1!} = \frac{10!}{3!3!2!1! \, 1!} = 50400$$

Combinations:

In a permutation, the order of the selected things is essential. In contrast, a combination of given things means any selection of one or more things without regard to order. There are two kinds of combinations, as follows.

The number of combinations of n different things, taken k at a time, without

repetitions is the number of sets that can be made up from the n given things, each set containing k different things and no two sets containing exactly the same k things. The number of combinations of n different things, taken k at a time, with repetitions is the number of sets that can be made up of k things chosen from the given n things, each being used as often as desired.

The number of different combinations of n different things taken, k at a time, without repetitions, is:

$$\binom{n}{k} = {}^{n}_{k}C = \frac{n!}{k!\,(n-k)!} = \frac{n(n-1)\,....\,(n-k+1)}{1.2\,....\,k}$$

And the number of those combinations with repetitions is:

$$\binom{n}{k} = {}^{n}_{k}C = \binom{n+1-k}{k}$$

NOTE

$$0! = 1 , \qquad\qquad (n+1)! = (n+1)n!$$

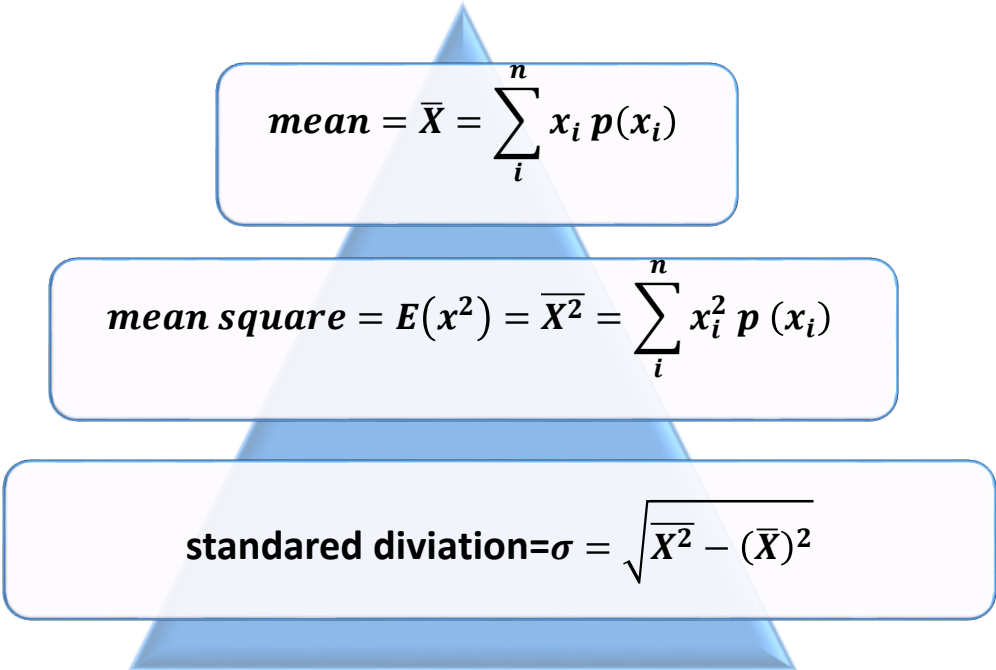$$\binom{n}{0} = 1, \qquad\qquad \binom{n}{n} = 1, \qquad \binom{n}{1} = n$$

Random Variables

Discrete random variable: a random variable X will be defined to be discrete if the range of X is countable. If a random variable X is discrete, then its corresponding cumulative distribution function F X (.) will be defined to be discrete, i.e. a step function.

$$\sum_{i}^{n} p(x_i) = 1$$

$$mean = \bar{X} = \sum_{i}^{n} x_i\, p(x_i)$$

$$mean\ square = E(x^2) = \overline{X^2} = \sum_{i}^{n} x_i^2\, p\,(x_i)$$

$$standared\ diviation = \sigma = \sqrt{\overline{X^2} - (\bar{X})^2}$$

EXAMPLE: An exam is taken by 9 students. After grading, it is found that the percentage scores are as listed below. Calculate the mean and standard deviation.

N	1	2	3	4	5	6	7	8	9
X	33	47	58	67	75	82	88	94	100

Sol:

$$\bar{X} = \sum_{i}^{n} x_i\, p(x_i)$$

$$p(x_i) = \frac{h}{n} = \frac{1}{9}$$

$$\bar{X} = 33 * \frac{1}{9} + 47 * \frac{1}{9} + 58 * \frac{1}{9} + 67 * \frac{1}{9} + 75 * \frac{1}{9} + 82 * \frac{1}{9} + 88 * \frac{1}{9} + 94 * \frac{1}{9} +$$

$$100 * \frac{1}{9} = 71.5$$

$$\overline{X^2} = \sum_i^n x_i^2 \, p\,(x_i) = (33)^2 * \frac{1}{9} + (47)^2 * \frac{1}{9} + (58)^2 * \frac{1}{9} + (67)^2 * \frac{1}{9} + (75)^2 *$$

$$\frac{1}{9} + (82)^2 * \frac{1}{9} + (88)^2 * \frac{1}{9} + (94)^2 * \frac{1}{9} + (100)^2 * \frac{1}{9} = 5564.4$$

$$\sigma = \sqrt{\overline{X^2} - (\overline{X})^2} = \sqrt{5564.4 - (71.5)^2} = \sqrt{452.15} = 21.26$$

Continuous Random Variable:

Continuous random variables appear in experiments in which we measure (lengths of screws, voltage in a power). A random variable X and its distribution are of continuous type or, briefly, continuous, if its distribution function can be given by an integrals:

$$P(x_1 < x < x_2) = \int_{x_1}^{x_2} p(x)dx$$

$$\int_{-\infty}^{\infty} p(x)dx = 1$$

$$\sigma = \sqrt{\overline{X^2} - (\overline{X})^2}$$

$$\overline{X} = \int_{-\infty}^{\infty} p(x) \, dx$$

$$\overline{X^2} = \int_{-\infty}^{\infty} x^2 \, p(x) \, dx$$

EXAMPLE: The random variable x has probability density function

$$p(x) = \begin{cases} cx & 0 \leq x \leq 2 \\ 0 & otherwise \end{cases}$$

1- Constant **c**

2- $p(0 \leq x \leq 1)$

3- $p\left(-\frac{1}{2} \leq x \leq \frac{1}{2}\right)$

4- $\bar{X}, \ \overline{X^2}, \sigma$

Sol:

1- $\int_{-\infty}^{\infty} p(x)dx = \int_{0}^{2} c \, x \, dx = 1$

$$\left.\frac{cx^2}{2}\right|_0^2 = 1 \ \rightarrow c = \frac{1}{2}$$

$$2 - p(0 \leq x \leq 1) = \int_0^1 c \, x \, dx = \frac{1}{2} \left.\frac{x^2}{2}\right|_0^1 = \frac{1}{4}(1) = \frac{1}{4}$$

$$3 - p\left(-\frac{1}{2} \leq x \leq \frac{1}{2}\right) = \int_0^{1/2} c \, x \, dx = \frac{1}{2} \left.\frac{x^2}{2}\right|_0^{1/2} = \frac{1}{4}\left(\frac{1}{4}\right) = \frac{1}{16}$$

$$4 - \bar{X} = \int_0^2 x \, p(x)dx = \int_0^2 cx^2 \, dx = \frac{1}{2}\int_0^2 x^2 \, dx = \frac{1}{2}\left.\frac{x^3}{3}\right|_0^2 = \frac{1}{2}\left(\frac{8}{3} - 0\right) = \frac{8}{6}$$

$$5 - \overline{X^2} = \int_0^2 x^2 \, p(x)dx = \int_0^2 cx^3 \, dx = \frac{1}{2}\int_0^2 x^3 \, dx = \frac{1}{2}\left.\frac{x^4}{4}\right|_0^2 = \frac{1}{2}\left(\frac{16}{4} - 0\right) = 2$$

$$\sigma = \sqrt{\overline{X^2} - (\bar{X})^2} = \sqrt{2 - \left(\frac{8}{6}\right)^2} = 0.471$$

Probability Distribution

Binomial Distribution: It is used for discrete events only.

$$(q + p)^n = \binom{n}{r} p^r q^{n-r}$$

$$= p^0 q^n + n\, pq^{n-1} + \frac{n(n-1)}{1*2}\, p^2\, q^{n-2} + \cdots$$

$$+ \frac{n(n-1)\dots(n-r+1)}{1*2*\dots r}\, p^r\, q^{n-r} + \cdots + p^n \binom{n}{n}$$

 Where:

n: number of attempts , **P**: probability of success , **q**: probability fail.

EXAMPLE: What is the probability of getting 0,1,2,3,4,…sixes in four throws of an unbiased dice?

Sol:

$$P(six) = \frac{1}{6}, \qquad q = 1 - p(not\ six) = 1 - 1/6 = 5/6$$

$$p\left(\frac{5}{6} + \frac{1}{6}\right) = \left(\frac{5}{6}\right)^4 + 4\left(\frac{1}{6}\right)^1 \left(\frac{5}{6}\right)^3 + \frac{4*3}{1*2}\left(\frac{1}{6}\right)^2 \left(\frac{5}{6}\right)^2 + \cdots + \frac{4*3*2}{1*2*3}\left(\frac{1}{6}\right)^3 \left(\frac{5}{6}\right) + \left(\frac{1}{6}\right)^4 \left(\frac{5}{6}\right)^0 =$$

$$\frac{171}{1296}$$

Statistics

Is that branch of mathematic that deals with the collection and analysis of data in the form of sets of values of discrete or continuous variable.

In order to evaluate the mean and standard deviation we apply the formula:

$$mean = \bar{x} = \sum_{i}^{n} \frac{f_i x_i}{n}$$

Where: f_i is the number of frequency.

n: the summation of frequencies.

The standard deviation is given by:

$$S = \sqrt{\frac{\sum_{i}^{n} f_i (x - \bar{x})^2}{n}}$$

EXAMPLE: The following results were obtained when measuring the diameters of 50 washers. Calculate the mean and standard deviation.

Diameter(x_i)	40	41	42	43	44
Number of washer(f_i)	7	12	18	8	5

Sol:

x_i	f_i	$(x - \bar{x})^2$	$f_i (x - \bar{x})^2$
40	7	3.24	22.68
41	12	0.64	7.68
42	18	0.04	0.72
43	8	1.44	11.52
44	5	4.84	24.2
	$\sum f_i = 50$		$\sum f_i (x - \bar{x})^2$ $= 66.8$

$$\bar{x} = \sum_{i}^{5} \frac{f_i x_i}{n} = \frac{7(40) + 12(41) + 18(42) + 8(43) + 5(44)}{7 + 12 + 18 + 8 + 5} = 41.8 \; mm$$

$$s = \sqrt{\frac{\sum_{i}^{n} f_i (x - \bar{x})^2}{n}} = \sqrt{\frac{66.8}{50}} = 1.15 \; mm$$

HOMEWORK

1- How many different samples of 6 objects can we draw from a lot of 24

a) With reputation   b) without reputation

3- In how many different ways can we select a committee consisting of 2 engineers, 1 physicists, and 5compute scientists from 7 engineers, 4 physicists, and 10 computer scientists?

4- Determine the mean and slandered deviation for the following data.

Degree	23	50	80	98	67
Number of student	1	8	2	3	5

5- Find the mean of variable x which has probability density distribution

$$p(x) = 4(2x^3 - 5x), \ \ 1 \le x \le 2$$

6- If the probability of Ali success in an exam is (0.3) and the probability of Ahmed failed in the same exam (0.2). Find the probability of Ali failed or Ahmed faild.

Chapter Four

Numerical Computations

Definition:

Is that branch of mathematical concerning of methods of obtaining numerical results for many problems.

Solution of Equation with One Variable

1- Bisection method.

If a continuous function $f(x)$ defined on the interval [a, b] with $f(a) < 0$ and $f(b) > 0$, then there exists c; $a < c < b$.

Steps for solution:

1- Compute $c_n = \frac{a_n + b_n}{2}$, $n = 0,1,2, \dots$

2- If $f(a)\, f(c) = 0$, set $a_{n+1} = c_n$ and $b_{n+1} = b_n$

3- The iteration is stopped if $\frac{c_{n-1} - c_n}{c_n} < \in$

    Where $\in$ is the tolerance.

EXAMPLE: $f(x) = x^3 + 4x^2 - 10$ has a root in [1, 2]. Use bisection method to find that root if $\epsilon = 10^{-4}$.

Sol:

$$c = \frac{a+b}{2} = \frac{1+2}{2} = 1.5$$

$$f(a_\circ) = f(1) = 1^3 + 4*1^2 - 10 = -5 \ (negetive)$$

$$f(b_\circ) = f(2) = 2^3 + 4*2^2 - 10 = 14 \quad (posetive)$$

a_n	b_n	c_n	$f(a_n) * f(b_n)$
1	2	1.5	- +
1	1.5	1.25	- -
1.25	1.5	1.375	- +
1.25	1.375	1.3125	- -
1.3125	1.375	1.343	- -
.			
.			
.			
1.364	1.365	1.3649	- -
1.3649	1.3652	1.365	- -

$$\left| \frac{c_{11} - c_{12}}{c_{12}} \right| \cong |-8.9 * 10^{-5}| < 10^{-4}$$

-**In order to calculate the number of iterations (n) we use the role:**

$$n \geq \frac{\log{(b-a)} - \log{\in}}{\log{2}}$$

EXAMPLE: Determine approximately how many iterations are necessary to solve

$f(x) = x^3 + 4x^2 - 10$ with accuracy of $\in = 10^{-5}$ in the interval [1,2].

Sol:

$$n \geq \frac{\log{(b-a)} - \log{\in}}{\log{2}} = \frac{\log{(2-1)} - \log{10^{-5}}}{\log{2}}$$

$$n \geq \frac{\log{1} + 5\log{10}}{\log{2}} = \frac{0 + 5*1}{0.301} = \frac{5}{0.301} = 16.6 \approx 17$$

2-Newton –Raphson method (N.R):

It is used for solving f (x) = 0, where f(x) has a continuous derivative $\acute{f}(x)$

Generally:

$$x_{n+1} = x_n - \frac{f(x_n)}{\acute{f}(x_n)} \quad , \acute{f}(x_n) \neq 0$$

EXAMPLE: Use Newton- Raphson method to solve f (x) where

$$f(x) = x^2 - 3x + 1 = 0, \quad x_0 = 2 .$$

Sol:

$$\acute{f}(x) = 2x - 3, n = 0, 1, 2, \dots$$

$$n = 0 \rightarrow \quad x_1 = x_0 - \frac{f(x_0)}{\acute{f}(x_0)} = 2 - \frac{2^2 - 3*2 + 1}{2*2 - 3} = 3$$

$$n = 1 \rightarrow \quad x_2 = x_1 - \frac{f(x_1)}{\acute{f}(x_1)} = 3 - \frac{3^2 - 3*3 + 1}{2*3 - 3} = 2.667$$

$$n = 2 \rightarrow \quad x_3 = x_2 - \frac{f(x_2)}{\acute{f}(x_2)} = 2.667 - \frac{-1.889}{2.334} = 2.619$$

Numerical Integration

Is the numerical evaluation of a definite integral.

$$A = \int_a^b f(x)\, dx$$

A is the area under the curve of f (x) between a, b.

1) Trapezoidal rule:

We divide the integral of integration into n equal of length $\quad h = \frac{b-a}{n}$
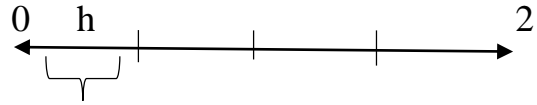
and approximate f (x) in each sub interval by a piecewise linear function, we obtain:

$$A = \int_a^b f(x)\, dx \cong h\left[\frac{1}{2} f(x_0) + f(x_1) + f(x_2) + \dots + f(x_{n-1}) + \frac{1}{2} f(x_n)\right]$$

EXAMPLE: Use trapezoidal method to evaluate $A = \int_0^2 x^2 \, dx$

   Sol:

$n = 4$, $h = \frac{b-a}{n} = \frac{2-0}{4} = 0.5$ $0 \quad h \qquad\qquad\qquad\qquad 2$

$$A = \int_a^b f(x) \, dx = \int_0^2 x^2 \, dx = h\,[\frac{1}{2}\,f(x_0) + f(x_1) + f(x_2) + f(x_3) + \frac{1}{2}f(x_4)]$$

$$A = 0.5\,[\frac{1}{2}\,f(0) + f(0.5) + f(1) + f(1.5) + \frac{1}{2}f(2)]$$

$$A = 0.5\,\left[\frac{1}{2} * 0 + 0.25 + 1 + 2.25 + \frac{1}{2} * 4\right] = 2.75$$

2) <u>Simpson's rule:</u>

We subdivide the interval of integration $a \le x \le b$ into an even number (2n) of equal subintervals.

$$h = \frac{b - a}{2n}$$

And f (x) is approximate using piecewise quadratic approximation. The result is:

$$A = \frac{h}{3}\,[G_0 + 4\,G_1 + 2\,G_2]$$

Where:

$$G_0 = f(a) + f(b)$$

$$G_1 = f(x_1) + f(x_3) + \cdots + f(x_{2n-1}), \qquad (odd\ values\ of\ x)$$

$$G_2 = f(x_2) + f(x_4) + \cdots + f(x_{2n-2}), \qquad (even\ values\ of\ x)$$

EXAMPLE: Evaluate $A = \int_0^1 e^{-x^2}\, dx$ using Simpson'srole with $2n = 10$.

Sol:

$$h = \frac{b-a}{2n} = \frac{1-0}{10} = 0.1$$

n	x_n	G_1	G_2
0	0	$f(x_1) = f(a) = f(0) = 1$	
1	0.1	0.99517	
2	0.2		0.96040
3	0.3	0.913931	
4	0.4		0.852144
5	0.5	0.778801	
6	0.6		0.697676
7	0.7	0.612626	
8	0.8		0.527292
9	0.9	0.444858	
10	1	$f(b) = f(2) = 0.367879$	

$$A = \frac{0.1}{3}[1 + 0.367879 + 4(3.740266) + 2(3.03791)] = 0.746825$$

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Numerical Differentiation

</div>

1- Difference formula:

$$\acute{f}(x_\circ) = \frac{f(x_\circ + h) - f(x_\circ)}{h} \qquad , h \neq 0$$

$x_1 = x_\circ + h$

If $h > 0$ *the formela is called forward.*

If $h < 0$ *the formela is called backward.*

EXAMPLE: If $f(x) = \ln x$, $x_\circ = 1.8$, find $\acute{f}(x_\circ)$ with $h = 0.1, \ 0.01 \ , 0.001$

Sol:

$$\acute{f}(x_\circ) = \frac{f(x_\circ + h) - f(x_\circ)}{h}$$

$$\acute{f}(1.8) = \frac{f(1.8 + 0.1) - f(1.8)}{0.1} = \frac{ln(1.9) - \ln(1.8)}{0.1}$$

$$= \frac{0.641 - 0.5877}{0.1} = \frac{0.0533}{0.1} = 0.533$$

$h = 0.01,$

$$\acute{f}(1.8) = \frac{f(1.8 + 0.01) - f(1.8)}{0.01} = \frac{ln(1.81) - \ln(1.8)}{0.01} = 0.554018$$

$h = 0.001,$

$$\acute{f}(1.8) = \frac{f(1.8 + 0.001) - f(1.8)}{0.001} = \frac{ln(1.801) - ln(1.8)}{0.001} = 0.555401$$

2- Three- points formula:

In this method, the points are equally spaced $x_0, \ x_1 = x_\circ + h, \ x_2 = x_1 + h$

Formula (1)

$$\acute{f}(x_\circ) = \frac{1}{2h} \left[-3f(x_\circ) + 4 \, f(x_1) - f(x_2) \right]$$

Formula (2)

$$\acute{f}(x_\circ) = \frac{1}{2h} \left[f(x_\circ + h) - f(x_\circ - h) \right]$$

Formula (2) has the advantage over formula (1) by half the error and evaluated only at two points.

EXAMPLE: If $f(x) = x \, e^x$, find $\acute{f}(2), h = 0.1$ using three point formulas.

   Sol:

   $\acute{f}(2) \ \ mean \ x_\circ = 2$

   $x_\circ - h = 1.9 , \quad x_1 = 2.1 , \quad x_2 = 2.2$

   Formula (1):

$$\acute{f}(x_\circ) = \frac{1}{2h} \left[-3f(x_\circ) + 4 f(x_1) - f(x_2) \right]$$

$$\acute{f}(x_\circ) = \frac{1}{2*0.1} \left[-3f(2) + 4 f(2.1) - f(2.2) \right] = 22.03231$$

Formula (2):

$$\acute{f}(x_\circ) = \frac{1}{2h} \left[f(x_\circ + h) - f(x_\circ - h) \right]$$

$$\acute{f}(2) = \frac{1}{2*0.1} \left[f(2 + 0.1) - f(2 - 0.1) \right] = 22.22879$$

HOMEWORK

1- Approximate $\acute{f}(1)$ for $f(x) = \sin x$ using three point formulas' with

$h = 0.5$

2- Solve the following equation using Newton Raphson method

$f(x) = 2x^2 + 3x = 0 \ \ with \ x_\circ = 0$

3- Evaluate $A = \int_0^1 \frac{1}{x^2+4} \ dx$ using:

   a) Trapezoidal method.

   b) Simpson's rule.