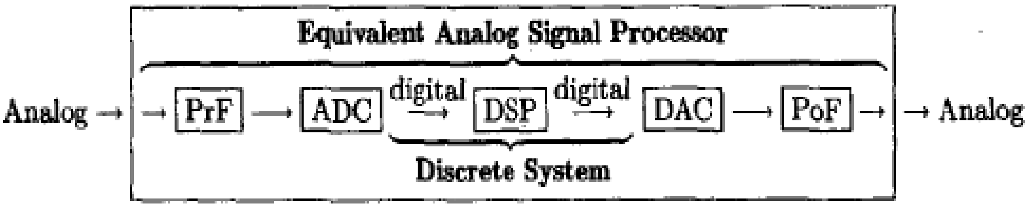
INTRODUCTION TO DIGITAL SIGNAL PROCESSING

1:General DSP system

The processing of digital signals is called DSP; in block diagram form it is represented by



Prf: This is a prefilter or an antialiasing filter, which conditions the analog signal to prevent aliasing.

ADC: This is called an analog-to-digital converter, which produces a stream of binary numbers from analog signals.

Digital signal processor: This is the heart of DSP and can represent a general-purpose computer or a special-purpose processor, or digital hardware, and so on.

DAC: This is the inverse operation to the ADC, called a digital-to-analog converter which produces a stair case waveform from a sequence of binary numbers, a first step towards producing an analog signal.

PoF: This is a postfilter to smooth out stair case waveform into the desired analog signal.

2-Drawback of analog signal processing (ASP)

A major drawback of ASP is its limited scope for performing complicated signal processing applications. This translates into nonflexibility in processing and complexity in system designs. All of these generally lead to expensive products.

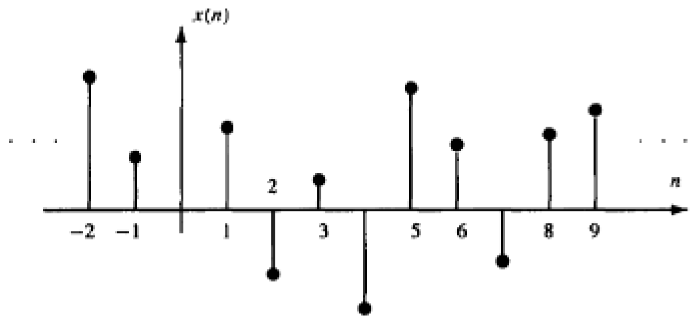
3. Advantages of DSP

1. System using the DSP approach can be developed using software running on a general-purpose computer. Therefore DSP is relatively convenient to develop and test, and the software is portable.
2. DSP operations are solely on additions and multiplication, leading to extremely stable processing capability-for example, stability independent of temperature.
3. DSP operations can easily be modified in real time, often by simple programming change, or by reloading of registers.
4. DSP has lower cost due to VLSI technology, which reduces costs of memories, gates, microprocessors, and so forth.

The principal disadvantage of DSP is the speed of operations, especially at very high frequencies. Primarily due to the above advantages, DSP is now becoming a first choice in many technologies and applications, such as consumer electronics, communications, wireless telephones, and medical imaging.

4.Discrete-Time Signals

A discrete-time signal is a function of an integer-valued variable, n, that is denoted by $x(n)$.



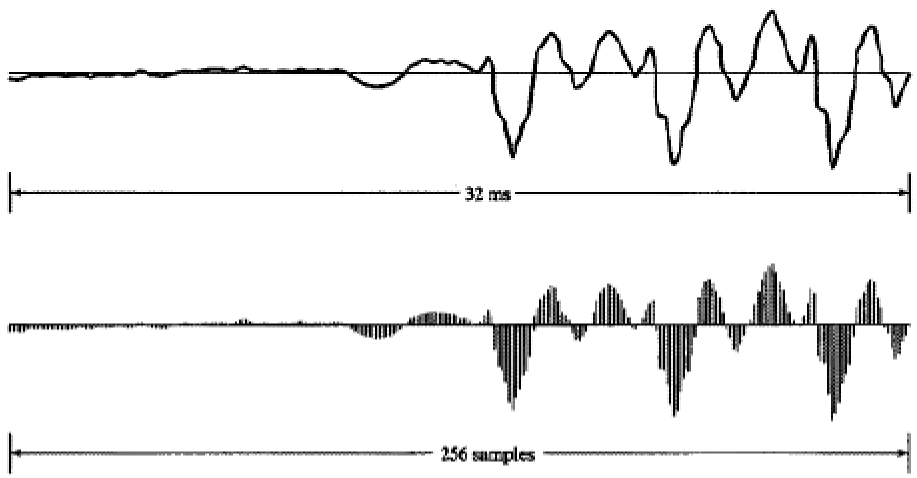
The sequence values $x(n)$ to $x(N-1)$ may often be considered to be the elements of a column vector as follows:

$$x = [x(0), x(1), \dots \dots, x(N-1)]^T \qquad (4.1)$$

Discrete-time signals are often derived by sampling a continuous-time signal, such as speech, with an analog to digital (A/D) converter. For example, a continuous-time signal $x_a(t)$ that is sampled at a rate of $f_s = 1/T_s(t)$ samples per second produces the sampled signal $x(n)$, which is related to $x_a(t)$ as follows:

$$x(n) = x_a(nT_s) \tag{4.2}$$

Figure below show a segment of a continuous-time speech signal and the sequence of samples that obtained from sampling it with $T_s = 125$ μs.



4.1 Complex Sequences

A complex signal may be expressed either in term of its real and imaginary parts.

$$z(n) = a(n) + jb(n) = \mathrm{Re}\{z(n)\} + \mathrm{jIm}\{z(n)\} = |z(n)| \exp[j \arg\{z(n)\}] \tag{4.3}$$

$$|z(n)|^2 = Re\{z(n)\}^2 + Im\{z(n)\}^2 \tag{4.4}$$

$$\arg\{z(n)\} = \tan^{-1} \frac{Im\{z(n)\}}{Re\{z(n)\}} \tag{4.5}$$

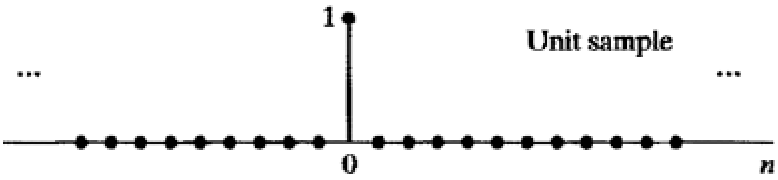
Also

$$z*(n) = a(n) - jb(n) = \mathrm{Re}\{z(n)\} - j\mathrm{Im}\{z(n)\} = |z(n)| \exp[-j \arg\{z(n)\}] \quad (4.6)$$

4.2 Some Fundamental Sequences

- **Unit sample $\boldsymbol{\delta(n)}$**

$$\delta(n) = \begin{cases} 1 & n = 0 \\ 0 & otherwise \end{cases} \quad (4.7)$$

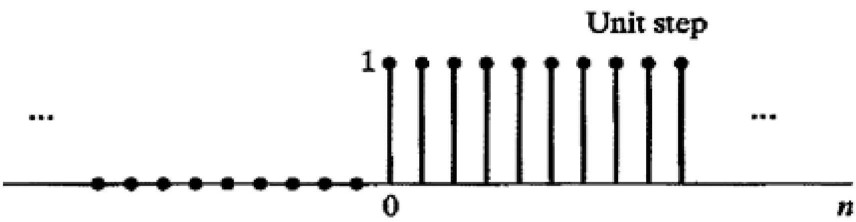


- **Unit step $\boldsymbol{u(n)}$**

$$u(n) = \begin{cases} 1 & n \geq 0 \\ 0 & otherwise \end{cases} \quad (4.8)$$

$$u(n) = \sum_{k=0}^{\infty} \delta(n-k) \quad (4.9)$$

$$\delta(n) = u(n) - u(n-1) \quad (4.10)$$



- **Exponential sequences**

$$x(n) = a^n \quad (4.11)$$

Where a may be a real or complex number. For real a and 0 < a < 1, $x(n)$ is shown below

If $x(n) = e^{j\omega_0}$, $x(n)$ is complex exponential.

$$x(n) = e^{jn\omega_0} = \cos(n\omega_0) + j\sin(n\omega_0) \tag{4.12}$$

4.3 Periodic and aperiodic Sequences

A signal $x(n)$ is said to be periodic if, for some positive real integer N

$$x(n) = x(n+N) \tag{4.13}$$

If $x_1(n)$ is a sequence that is periodic with a period N_1, and $x_2(n)$ is another sequence that is periodic with a period N_2, the sum $x(n) = x_1(n) + \ x_2(n)$ or the product $x(n) = x_1(n)\ x_2(n)$ will always be periodic and the fundamental period is

$$N = \frac{N_1 N_2}{\mathrm{g}cd(N_1,N_2)} \tag{4.14}$$

Where $\mathrm{g}cd(N_1, N_2)$ means ***the greatest common divisor*** of N_1 and, N_2

4.4 Symmetric Sequences

$$x(n) = x(-n) \qquad \rightarrow even \tag{4.15}$$

$$x(n) = -x(-n) \qquad \rightarrow odd \tag{4.16}$$

$$x(n) = x*(-n) \qquad \rightarrow conjugate\ symmetric \tag{4.17}$$

$$x(n) = -x*(-n) \rightarrow conjugate\ antisymmetric \tag{4.18}$$

Any signal $x(n)$ may be decomposed into a sum of its even part, $x_e(n)$, and its odd part, $x_o(n)$, as follows:

$$x(n) = x_e(n) + x_o(n) \tag{4.19}$$

Where the even part of a signal $x(n)$ is given by

$$x_e(n) = \frac{1}{2}[x(n) + x(-n)] \tag{4.20}$$

Where the odd part of a signal $x(n)$ is given by difference

$$x_o(n) = \frac{1}{2}[x(n) - x(-n)] \tag{4.21}$$

The conjugate symmetric part of $x(n)$ is

$$x_e(n) = \frac{1}{2}[x(n) + x^*(-n)] \tag{4.22}$$

The conjugate antisymmetric part of $x(n)$ is

$$x_o(n) = \frac{1}{2}[x(n) - x^*(-n)] \tag{4.23}$$

4.5 Transformations of the Independent Variable

$$y(n) = x(f(n)) \tag{4.24}$$

Where $f(n)$ is some function of n. The most common transformations are:

- **Shifting:** If $y(n) = x(n - n_0)$, $x(n)$ is shifted to the right by n_0 samples if n_0 is positive (this is referred to as delay), and it is shifted to the left by n_0 samples if n_0 is negative (referred to as an advance).
- **Reversal:** This transformation is given by $f(n) = -n$ and simple involves "flipping" the signal $x(n)$ with respect to the index n.
- **Time Scaling:**
  - **Down-Sampling:** $f(n) = Mn$ the sequence $x(Mn)$ is formed by taking every M^{th}sample of $x(n)$.
  - **Up-Sampling:** $f(n) = n/N$ the sequence $y(n) = x(f(n))$ is defined as follows:

$$y(n) = \begin{cases} x\left(\frac{n}{N}\right) & n = 0, \pm N, \pm 2N, \ldots. \\ 0 & otherwise \end{cases}$$

4.6 Addition, Multiplication, and Scaling

- **Addition:** $y(n) = x_1(n) + x_2(n) \qquad -\infty < n < \infty$
- **Multiplication:** $y(n) = x_1(n)\, x_2(n) \qquad -\infty < n < \infty$
- **Scaling:** $y(n) = cx(n) \qquad -\infty < n < \infty$

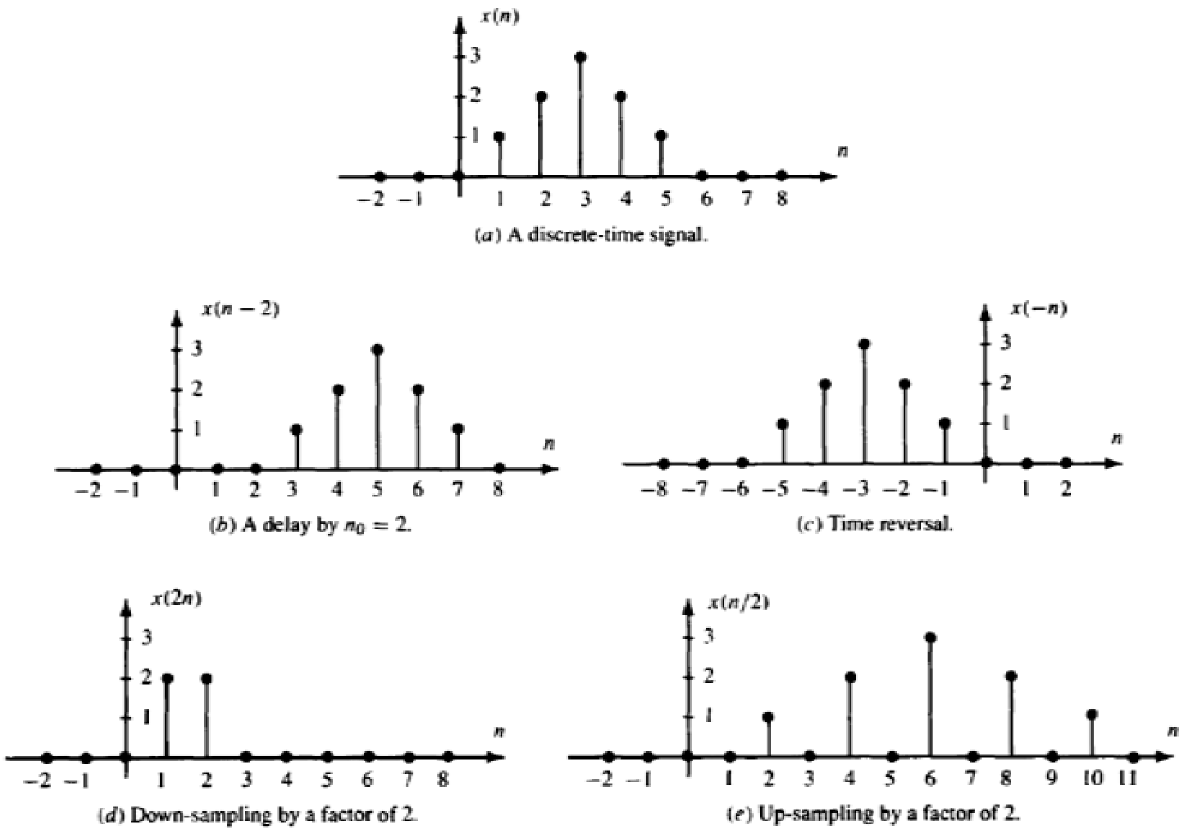
4.7 Signal Decomposition

The unit sample may be used to decompose an arbitrary signal $x(n)$ into a sum of weighted and shifted unit samples as follows:

$$x(n) = \cdots + x(-1)\delta(n+1) + x(0)\delta(n) + x(1)\delta(n-1) + x(2)\delta(n-2) + \cdots$$

$$x(n) = \sum_{k=-\infty}^{\infty} x(k)\delta(n-k) \tag{4.25}$$

Example 4.1:



(a) A discrete-time signal.

(b) A delay by $n_0 = 2$.

(c) Time reversal.

(d) Down-sampling by a factor of 2.

(e) Up-sampling by a factor of 2.

Example 4.2: Given the sequence

$$x(n) = 2\delta(n+2) + \delta(n+1) + 2\delta(n) + 4\delta(n-1) + \delta(n-2)\,,$$

make a sketch of:

a) $y_1(n) = x(n-2)$

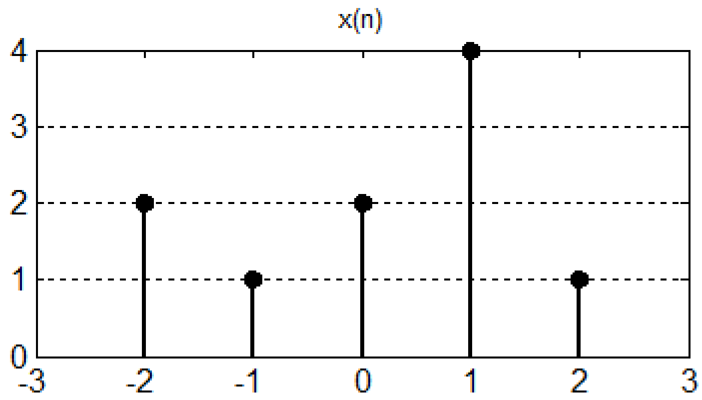
b) $y_2(n) = x(n+3)$

c) $y_3(n) = 2x(n-1)$

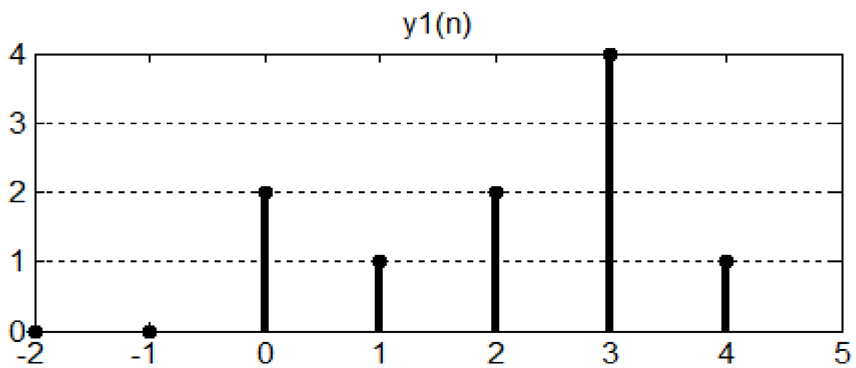
d) $y_4(n) = x(-n)$

Solution:

$$x(n) = 2\delta(n+2) + \delta(n+1) + 2\delta(n) + 4\delta(n-1) + \delta(n-2)$$

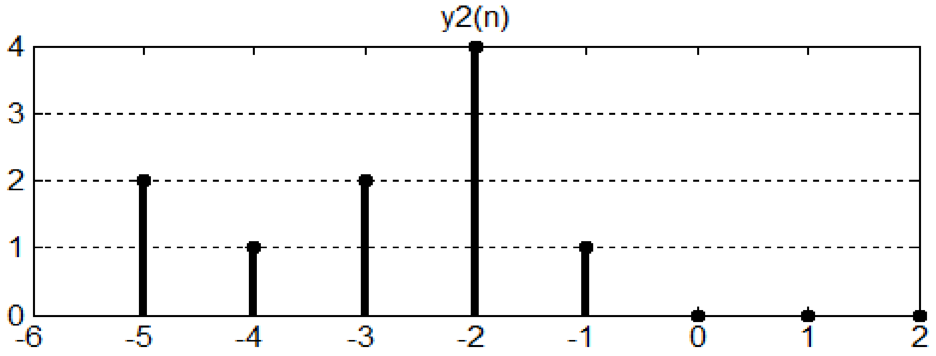


a) $y_1(n) = x(n-2)$



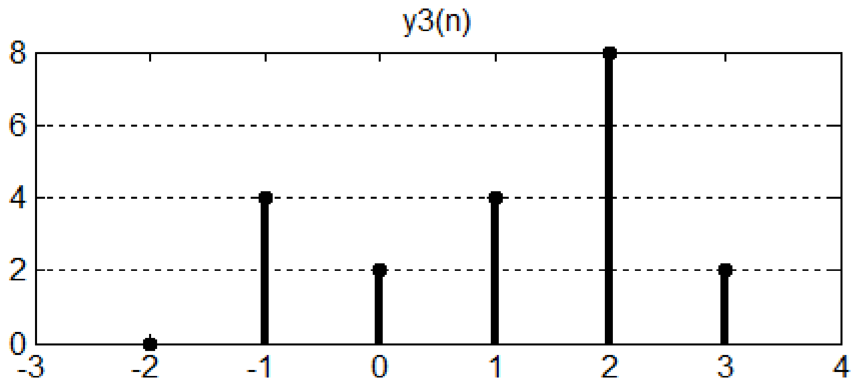
(a)

b) $\boldsymbol{y_2(n) = x(n+3)}$



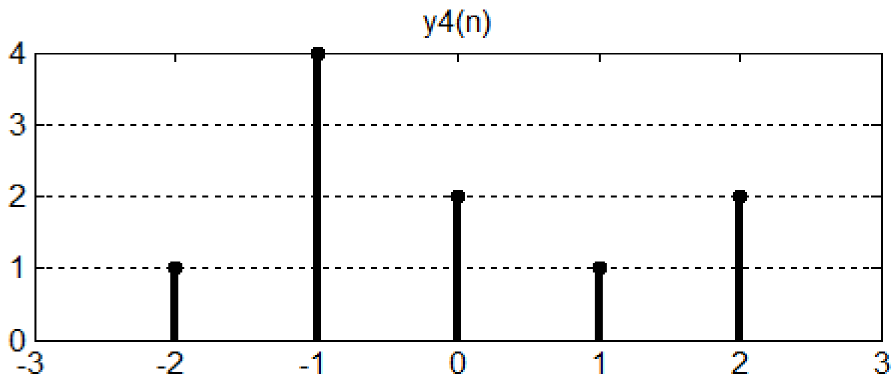
(b)

c) $\boldsymbol{y_3(n) = 2x(n-1)}$



(c)

d) $\boldsymbol{y_4(n) = x(-n)}$



(d)

Example 4.3: Given the sequence $x(n) = (6-n)[u(n)-u(n-6)]$, make a sketch of:

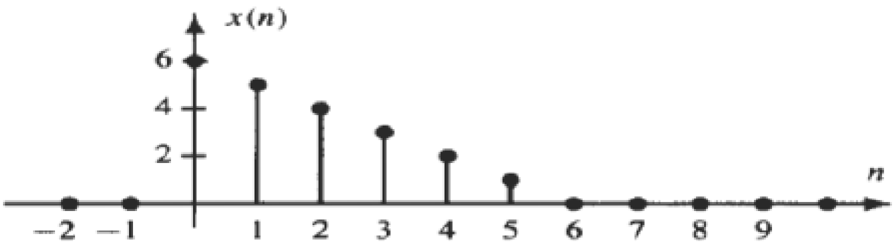
a) $\boldsymbol{y_1(n) = x(4-n)}$

b) $\boldsymbol{y_2(n) = x(2n-3)}$

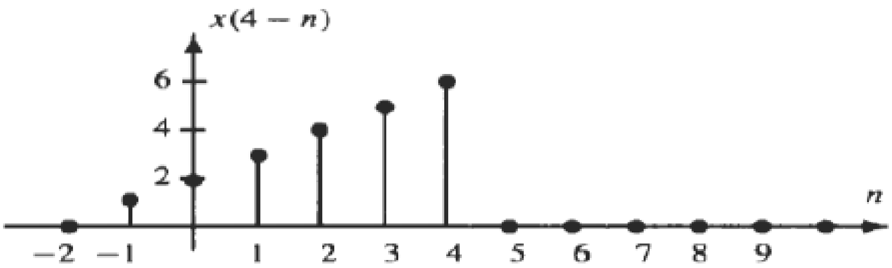
c) $\boldsymbol{y_3(n) = x(8-3n)}$

Solution:

$$\boldsymbol{x(n) = (6-n)[u(n)-u(n-6)]}$$

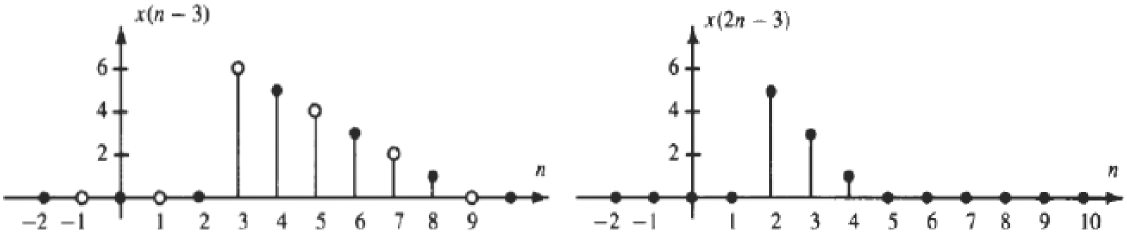


a) $\boldsymbol{y_1(n) = x(4-n)}$



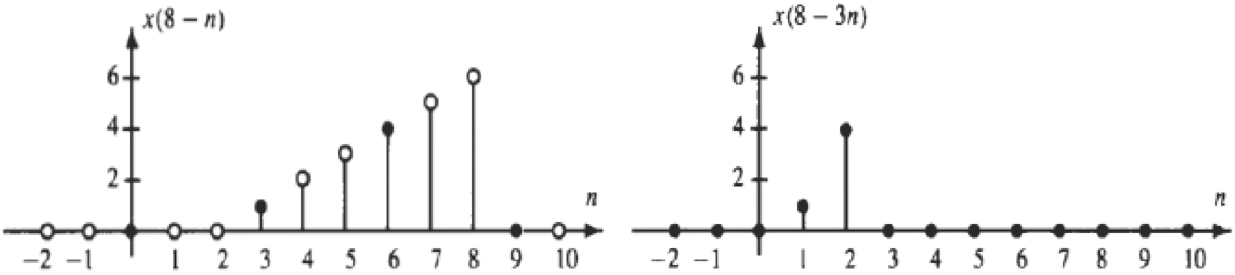
(a)

b) $\boldsymbol{y_2(n) = x(2n-3)}$



(b)

c) $y_3(n) = x(8 - 3n)$



(c)

<u>Example 4.4:</u> A discrete-time signal $x(n) = [\,1\ \ \bar{1}\ \ 1\ \ 1\ 1\ \ \frac{1}{2}\,]$,

Sketch each of the following signals:

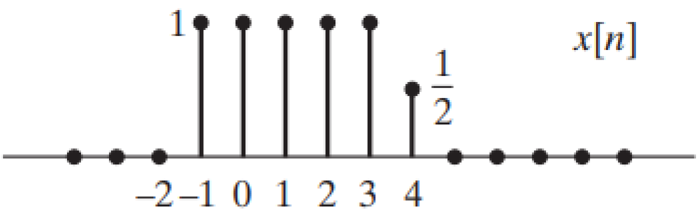
(a) x[n − 2]

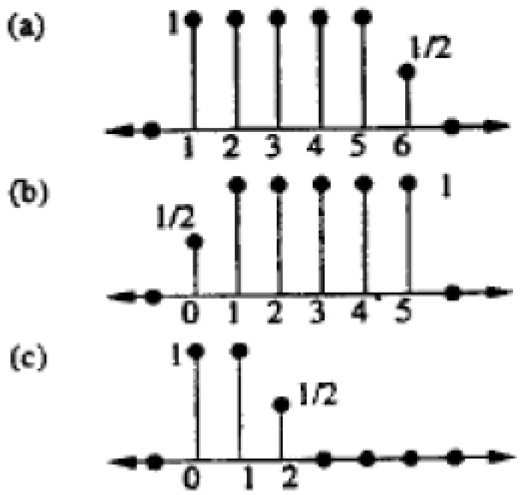
(b) x[4 − n]

(c) x[2n]

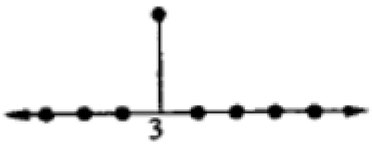
(d) x[n − 1]δ[n − 3].

<u>Solution:</u>



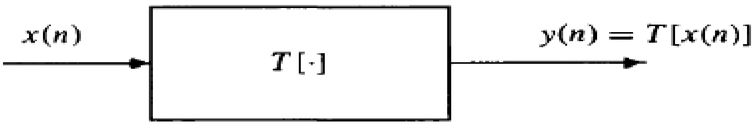


(d)



5. Discrete-Time System

In a discrete-time system an input signal $x(n)$ is transformed into an output signal $y(n)$ through the transformation $T[\cdot]$



5.1 System Properties

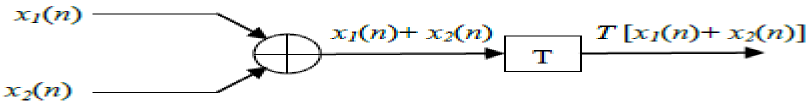
- ***Memoryless System*:** a system is memoryless if, for any n_0 , we are able to determine the value of $y(n_0)$ given only the value of $x(n_0)$.

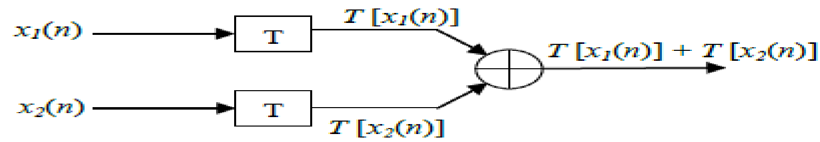
  Example 5.1:

  1- $y(n) = x^2(n)$ is memoryless

  2- $y(n) = x(n) + x(n-1)$ is not memoryless.

- ***Additivity:*** $\boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n}) + \boldsymbol{x_2}(\boldsymbol{n})] = \boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n})] + \boldsymbol{T}[\boldsymbol{x_2}(\boldsymbol{n})]$





<u>Example 5.2:</u> The system defined by $y(n) = \frac{x^2(n)}{\mathrm{x}(n-1)}$ is additive

<u>Solution:</u> -

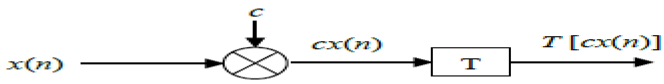
$$\boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n}) + \boldsymbol{x_2}(\boldsymbol{n})] = \frac{(x_1(n) + x_2(n))^2}{x_1(n-1) + x_2(n-1)}$$

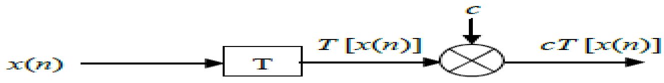
$$\boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n})] + \boldsymbol{T}[\boldsymbol{x_2}(\boldsymbol{n})] = \frac{{x_1}^2(n)}{x_1(n-1)} + \frac{{x_2}^2(n)}{x_2(n-1)}$$

The system $y(n) = \frac{x^2(n)}{\mathrm{x}(n-1)}$ is not additive because

$$\boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n}) + \boldsymbol{x_2}(\boldsymbol{n})] \neq \boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n})] + \boldsymbol{T}[\boldsymbol{x_2}(\boldsymbol{n})]$$

- ***Homogeneity*: $\boldsymbol{T}[\boldsymbol{cx}(\boldsymbol{n})] = \boldsymbol{cT}[\boldsymbol{x}(\boldsymbol{n})]$**





<u>Example 5.3:</u> The system defined by $y(n) = \frac{x^2(n)}{\mathrm{x}(n-1)}$

$$\boldsymbol{T}[\boldsymbol{cx}(\boldsymbol{n})] = \frac{(cx(n))^2}{cx(n-1)} = \frac{cx^2(n)}{x(n-1)}$$

$$\boldsymbol{cT}[\boldsymbol{x}(\boldsymbol{n})] = c\frac{x^2(n)}{x(n-1)} = \frac{cx^2(n)}{x(n-1)}$$

$$\boldsymbol{T}[\boldsymbol{cx}(\boldsymbol{n})] = \boldsymbol{cT}[\boldsymbol{x}(\boldsymbol{n})]$$

This system is, homogeneous

Example 5.4: the system defined by the equation

$$y(n) = x(n) + x^*(n-1)$$

$$\boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n}) + \boldsymbol{x_2}(\boldsymbol{n})] = [x_1(n) + x_2(n)] + [x_1(n-1) + x_2(n-1)]^*$$

$$= [x_1(n) + {x_1}^*(n-1)] + [x_2(n) + {x_2}^*(n-1)]$$

$$\boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n})] + \boldsymbol{T}[\boldsymbol{x_2}(\boldsymbol{n})] = [x_1(n) + {x_1}^*(n-1)] + [x_2(n) + {x_2}^*(n-1)]$$

Is additive because

$$\boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n}) + \boldsymbol{x_2}(\boldsymbol{n})] = \boldsymbol{T}[\boldsymbol{x_1}(\boldsymbol{n})] + \boldsymbol{T}[\boldsymbol{x_2}(\boldsymbol{n})]$$

for homogeneous *Property* he response to $cx(n)$ is

$$\boldsymbol{T}[\boldsymbol{cx}(\boldsymbol{n})] = cx(n) + c^*x^*(n-1)$$

$$\boldsymbol{cT}[\boldsymbol{x}(\boldsymbol{n})] = cx(n) + cx^*(n-1)$$

this is not homogeneous because the response to $cx(n)$ is

$$\boldsymbol{T}[\boldsymbol{cx}(\boldsymbol{n})] \neq \boldsymbol{cT}[\boldsymbol{x}(\boldsymbol{n})]$$

- ***Linear System:*** A system that is both additive and homogeneous is said to be linear. Thus,

$$T[a_1x_1(n) + a_2x_2(n)] = a_1T[x_1(n)] + a_2T[x_2(n)] \qquad (5.1)$$

- ***Shift-Invariance***

Let $y(n) = T[x(n)]$, then if $y(n-n_o) = T[x(n-n_o)]$, the system is considered as time invariant system. A system that is not shift-invariant is said to be shift-varying.

$$x(n) \longrightarrow \boxed{T} \xrightarrow{y(n)} \boxed{n_o} \xrightarrow{y(n-n_o)}$$

$$x(n) \longrightarrow \boxed{n_o} \xrightarrow{x(n-n_o)} \boxed{T} \xrightarrow{T[x(n-n_o)]}$$

Example 5.6: to test system defined by $y(n) = x^2(n)$ is shift-invariant.

$$y(n) = [x(n)]^2 \rightarrow y(n-n_o) = x^2(n-n_o)$$

$$y'(n) = [x'(n)]^2 = [x(n-n_o)]^2 = x^2(n-n_o)$$

Because $y(n-n_o) = y'(n)$, the system is shift-invariant.

Example 5.7: the system described by the equation $y(n) = x(n) + x(-n)$ is shift-varying.

Solution: The shifted output $y(n - n_o) = x(n - n_o) + x(-(n - n_o))$

$x'(n) = x(n - n_o)$,then

$y'(n) = x'(n) + x'(-n) = x(n - n_o) + x(-n - n_o)$.

. Because $y(n - n_o) \neq y'(n)$, the system is shift-varying.

- ***Linear shift-Invariant System***: A system that is both linear and shift-invariant is referred to as a linear shift-invariant (LSI) system. The output $y(n)$ is given by

$$y(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \tag{5.2}$$

Which is known as the convolution sum $y(n) = x(n) * h(n)$

- ***Causality*****:** A system to be causal if, for any n_o **,** the response of the system at time n_o depends only on the input up to time $n = n_o$. An LSI system will be causal if and only if $h(n)$ is equal to zero for $n < 0$.

Example 5.5: the system described by the equation $y(n) = x(n) + x(n - 1)$ is causal because the value of the output at any time $n = n_o$ depends only on the input $x(n)$ at n_o and at time $n_o - 1$.

Example 5.6:The system described by $y(n) = x(n) + x(n + 1)$, is noncausal because the output at time $n = n_o$ depends on the value of the input at time $n_o + 1$.

- ***Stability*****:** A system is said to be stable in the bounded input-bounded output (BIBO) sense if, for any input that is bounded, $|x(n)| \leq A \leq \infty$, the output will be bounded, $|y(n)| \leq B \leq \infty$

For a LSI system, stability is guaranteed if the unit sample response is absolutely summable:

$$\sum_{k=-\infty}^{\infty} |h(n)| < \infty \tag{5.3}$$

Example 5.7: an LSI system with unit sample response $h(n) = a^n u(n)$ will be stable whenever $|a| < 1$, because

$$\sum_{n=-\infty}^{\infty} |h(n)| = \sum_{n=0}^{\infty} |a|^n = \frac{1}{1-|a|} \qquad |a| < 1$$

Example 5.8: The system described by the equation $y(n) = nx(n)$, is not stable because the response to a unit step, $x(n) = u(n)$, is $y(n) = nu(n)$, which is unbounded.

The *z*-transform

The *z*-transform of a discrete-time signal *x*[*n*] is defined by

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n}$$

Where $z = re^{jw}$ is a complex variable. The values of z for which the sum converges define a region in the z-plane referred to as the region of convergence (ROC). If $x(n)$ has a z-transform $X(z)$, we write

$$x(n) \overset{z}{\leftrightarrow} X(z)$$

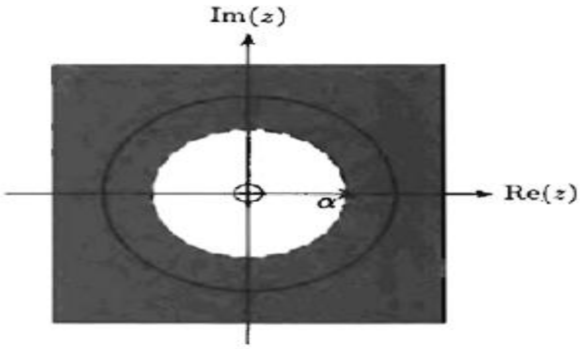
Example 6.1: Find the z-transformer of the sequence $x(n) = a^n u(n)$.

Solution: Using the definition of the z-transform and geometric series given in table ,we have

$$X(Z) = \sum_{n=-\infty}^{\infty} x(n) z^{-n} = \sum_{n=-\infty}^{\infty} a^n u(n) z^{-n} = \sum_{n=0}^{\infty} a^n z^{-n} = \sum_{n=0}^{\infty} (az^{-1})^n$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

Considered the z-transform of a right-sided sequence, which led to a region of convergence that is the exterior of a circle. ROC $|z|>|a|$



Sequence	z-Transform	Region of Convergence
$\delta(n)$	1	all z
$\alpha^n u(n)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| > \|\alpha\|$
$-\alpha^n u(-n-1)$	$\dfrac{1}{1-\alpha z^{-1}}$	$\|z\| < \|\alpha\|$
$n\alpha^n u(n)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| > \|\alpha\|$
$-n\alpha^n u(-n-1)$	$\dfrac{\alpha z^{-1}}{(1-\alpha z^{-1})^2}$	$\|z\| < \|\alpha\|$
$\cos(n\omega_0)u(n)$	$\dfrac{1-(\cos\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$
$\sin(n\omega_0)u(n)$	$\dfrac{(\sin\omega_0)z^{-1}}{1-2(\cos\omega_0)z^{-1}+z^{-2}}$	$\|z\| > 1$

<u>Example 6.2:</u> find the z-transform of the sequence $x(n) = -a^n u(-n-1)$.

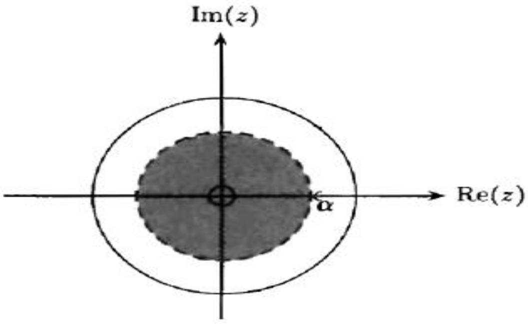
<u>Solution:</u> Proceeding as in the previous example, we have

$$X(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} -a^n u(-n-1)z^{-n} = -\sum_{n=-\infty}^{1} a^n z^{-n}$$

$$= -\sum_{n=0}^{\infty} a^{-n-1}z^{n+1} = -\sum_{n=0}^{\infty}(a^{-1}z)^{n+1} = -a^{-1}z\sum_{n=0}^{\infty}(a^{-1}z)^n = -\frac{a^{-1}z}{1-a^{-1}z}$$

$$= \frac{1}{1-az^{-1}} = \frac{z}{z-a}$$

ROC $|z| < |a|$



Note: Comparing the z-transforms of the signals in Example 6.1 and 6.2, we see that they are the same, differing only in their regions of convergence. Thus, the z-transform of a sequence is not uniquely defined until its regions of convergence has been specified.

Example 6.3: Find the z-transform of $x(n) = \left(\frac{1}{2}\right)^n u(n) - (2)^n u(-n-1)$, and find another signal that has the same z-transform but a different region of convergence.

Solution: Here we have a sum of two sequences. Therefore, we may find the z-transform of each sequence separately and add them together. From Example 6.1, we know that the z-transform of $x_1(n) = \left(\frac{1}{2}\right)^n u(n)$ is

$$X_1(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} \left(\frac{1}{2}\right)^n u(n)z^{-n} = \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^n z^{-n}$$

$$= \sum_{n=0}^{\infty} \left(\left(\frac{1}{2}\right) z^{-1}\right)^n$$

$$X_1(z) = \frac{1}{1-\frac{1}{2}z^{-1}} \quad |z| > \left|\frac{1}{2}\right|$$

And from Example 6.2 that the z-transform of $x_2(n) = -2^n u(-n-1)$

$$X_2(z) = \sum_{n=-\infty}^{\infty} x(n)z^{-n} = \sum_{n=-\infty}^{\infty} -2^n u(-n-1)z^{-n} = -\sum_{n=-\infty}^{1} 2^n z^{-n}$$

$$= -\sum_{n=0}^{\infty} 2^{-n-1} z^{n+1} = -\sum_{n=0}^{\infty} (2^{-1}z)^{n+1} = -2^{-1}z \sum_{n=0}^{\infty} (2^{-1}z)^{n} = -\frac{2^{-1}z}{1 - 2^{-1}z}$$

$$= \frac{1}{1 - 2z^{-1}}$$

$$X_2(z) = \frac{1}{1-2z^{-1}} \quad |z| < |2|$$

Therefore, the z-transform of $x(n) = x_1(n) + x_2(n)$ is

$$X(z) = X_1(z)+X_2(z)$$

$$X(z) = \frac{1}{1-\frac{1}{2}z^{-1}} + \frac{1}{1-2z^{-1}} = \frac{2-\frac{5}{2}z^{-1}}{(1-\frac{1}{2}z^{-1})(1-2z^{-1})}$$

With a region of convergence $\frac{1}{2} < z < 2,$ which is the set of all points that are in the ROC of both $X_1(z)$ and $X_2(z)$.

To find another sequence that has the same z-transform, note that because $X(z)$ is a sum of two z-transforms,

$$X(z) = \frac{1}{1 - \frac{1}{2}z^{-1}} + \frac{1}{1 - 2z^{-1}}$$

Each term corresponds to the z-transform of either a right-sided or a left –sided sequence, depending upon the region of convergence. Therefore, choosing the right-side sequences for both terms, it follows that

$$\acute{x}(n) = \frac{1}{2}^{n} u(n) + 2^{n}u(n)$$

Has the same z-transform as$x(n)$, except that the region of convergence is $|z| > |2|$.

6.1 Properties of z-transform

Property	Sequence	z-Transform	Region of Convergence
Linearity	$ax(n)+by(n)$	$aX(z)+bY(z)$	Contains $R_x \cap R_y$
Shift	$x(n-n_0)$	$z^{-n_0}X(z)$	R_x
Time reversal	$x(-n)$	$X(z^{-1})$	$1/R_x$
Exponentiation	$\alpha^n x(n)$	$X(\alpha^{-1}z)$	$\lvert\alpha\rvert R_x$
Convolution	$x(n) * y(n)$	$X(z)Y(z)$	Contains $R_x \cap R_y$
Conjugation	$x^*(n)$	$X^*(z^*)$	R_x
Derivative	$nx(n)$	$-z\dfrac{dX(z)}{dz}$	R_x

1.Time Shifting

If $x(n) \overset{Z.T}{\leftrightarrow} X(z)$ then $x(n-n_o) \overset{Z.T}{\leftrightarrow} Z^{-n_o}X(z)$

Proof: $x(n-n_o) \overset{Z.T}{\leftrightarrow} Z^{-n_o}X(z)$

$$Z\{x(n-n_o)\} = \sum_{n=-\infty}^{\infty} x(n-n_o)\, Z^{-n}$$

Assume that $k = n - n_o \rightarrow n = k + n_o$

$$\therefore Z\{x(n-n_o)\} = \sum_{k=-\infty}^{\infty} x(k)\, Z^{-(k+n_o)} = \sum_{k=-\infty}^{\infty} x(k)\, Z^{-k-n_o}$$

$$= Z^{-n_o} \sum_{k=-\infty}^{\infty} x(k)\, Z^{-k}$$

$$= Z^{-n_o}X(z)$$

2. Multiplication by an Exponential Sequence

$y(n) = Z_o^n x(n)$ then $Y(z) = X(\frac{z}{z_o})$

Proof: $y(n) = Z_o^n x(n)$ then $Y(z) = X(\frac{z}{z_o})$

$$Y(Z) = \sum_{n=-\infty}^{\infty} y(n) z^{-n}$$

$$Y(Z) = \sum_{n=-\infty}^{\infty} Z_o^n x(n)\, z^{-n} = \sum_{n=-\infty}^{\infty} x(n) \left(\frac{z}{z_o}\right)^{-n} = X\left(\frac{z}{z_o}\right)$$

3.Differentiation of F(z)

$f(n) \overset{Z.T}{\leftrightarrow} F(z)$ then $nf(n) \overset{Z.T}{\leftrightarrow} -z\frac{dF(z)}{dz}$

Proof: $f(n) \overset{Z.T}{\leftrightarrow} F(z)$ then $nf(n) \overset{Z.T}{\leftrightarrow} -z\frac{dF(z)}{dz}$

$$F(z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$-z\frac{dF(z)}{dz} = -z \sum_{n=-\infty}^{\infty} -nf(n) z^{-n-1} = -z \sum_{n=-\infty}^{\infty} -nf(n) z^{-n} z^{-1}$$

$$= \sum_{n=-\infty}^{\infty} nf(n) z^{-n}$$

$$nf(n) \overset{Z.T}{\leftrightarrow} -z\frac{dF(z)}{dz}$$

4. Conjugation of a complex sequence

$f(n) \overset{Z.T}{\leftrightarrow} F(z)$ then $f^*(n) \overset{Z.T}{\leftrightarrow} F^*(z^*)$

Proof: $f(n) \overset{Z.T}{\leftrightarrow} F(z)$ then $f^*(n) \overset{Z.T}{\leftrightarrow} F^*(z^*)$

$$F(Z) = \sum_{n=-\infty}^{\infty} f(n) z^{-n}$$

$$Z\{f^*(n)\} = \sum_{n=-\infty}^{\infty} f^*(n)\, z^{-n}$$

$$= \left(\sum_{n=-\infty}^{\infty} f(n)(z^*)^{-n}\right)^*$$

$$= F^*(z^*)$$

5.Time Reversal

$f(-n) \overset{Z.T}{\leftrightarrow} F\left(\frac{1}{z}\right)$

Proof: $f(-n) \overset{Z.T}{\leftrightarrow} F\left(\frac{1}{z}\right)$

$$Z\{f(-n)\} = \sum_{n=-\infty}^{\infty} f(-n)\, Z^{-n}$$

Let $k = -n$

$$= \sum_{k=-\infty}^{\infty} f(k)\, Z^{k} = \sum_{k=-\infty}^{\infty} f(k)\, \left(\frac{1}{z}\right)^{-k}$$

$$= F\left(\frac{1}{z}\right)$$

6.Convolution of Sequences

$x_1(n) * x_2(n) \overset{Z.T}{\leftrightarrow} X_1(z)X_2(z)$,ROC $R_{x1} \cap R_{x2}$

Let

$$y(n) = \sum_{n=-\infty}^{\infty} x_1(k)\, x_2(n-k)$$

So that

$$Y(Z) = \sum_{k=-\infty}^{\infty} y(n)\, z^{-n}$$

$$Y(Z) = \sum_{n=-\infty}^{\infty} \{ \sum_{k=-\infty}^{\infty} x_1(k)\, x_2(n-k) \}\, z^{-n}$$

If we interchange the order of summation

$$Y(Z) = \sum_{k=-\infty}^{\infty} x_1(k) \sum_{n=-\infty}^{\infty} x_2(n-k)\, z^{-n}$$

Let $m = n - k$

$$Y(Z) = \sum_{k=-\infty}^{\infty} x_1(k) \{ \sum_{m=-\infty}^{\infty} x_2(m)\, z^{-m} \} z^{-k}$$

$$Y(Z) = \sum_{k=-\infty}^{\infty} x_1(k) z^{-k} \{ \sum_{m=-\infty}^{\infty} x_2(m)\, z^{-m} \}$$

$$Y(Z) = X_1(Z).X_2(Z)$$

Example 6.4: Find the z-transform of $x(n) = na^n u(-n)$.

Solution: To find $X(z)$, we will use the time-reversal and derivative properties. First, as we saw in Example 6.1 ,

$(a)^n u(n) \overset{Z}{\leftrightarrow} \frac{1}{1-az^{-1}}$ ROC $|z| > a$

Therefore,

$(a^{-1})^n u(n) \overset{Z.T}{\leftrightarrow} \frac{1}{1-a^{-1}z^{-1}}$ ROC $|z| > \frac{1}{a}$

And, using the time-reversal property,

$(a)^n u(-n) \overset{Z}{\leftrightarrow} \frac{1}{1-a^{-1}z}$ ROC $|z| < a$

Finally, using the derivative property

$$-z\frac{d}{dz}\frac{1}{1-a^{-1}z} = -\frac{a^{-1}z}{(1-a^{-1}z)^2}$$ ROC $x(n) = u(n) - (0.5)^n u(n).$ $|z| < a$

Example 6.5: Find the z-transform of $x(n) = u(n) - (0.5)^n u(n)$.

Solution:

Applying the linearity of the z-transform, we have

$$X(z) = Z(x(n)) = Z(u(n)) - Z(0.5^n(n)).$$

$$Z(u(n)) = \frac{z}{z-1}$$

and $Z(0.5^n u(n)) = \frac{z}{z-0.5}$.

Substituting these results into X(z) leads to the final solution,

$$X(z) = \frac{z}{z-1} - \frac{z}{z-0.5}$$

Example 6.5: Find the z-transform of the following sequence:

$$y(n) = (0.5)^{(n-5)} \cdot u(n-5),$$

where $u(n-5) = 1$ for $n \geq 5$ and $u(n-5) = 0$ for $n < 5$.

Solution:

We first use the shift theorem to have

$$Y(z) = Z\left[(0.5)^{n-5}u(n-5)\right] = z^{-5}Z[(0.5)^n u(n)].$$

$$Y(z) = z^{-5} \cdot \frac{z}{z-0.5} = \frac{z^{-4}}{z-0.5}.$$

6.2 Partial Fraction Expansion

For z-transforms that are rational functions of z, a simple and straightforward approach to find the inverse z-transform is to perform a partial fraction expansion of $X(z)$.

Example 6.6: Find the inverse z-transform of the following $X(z)$.

$$X(z) = \frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{1 - \frac{3}{4}z^{-1} + \frac{1}{8}z^{-2}} = \frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

With a region of convergence $|z| > \frac{1}{2}$.

Solution: the partial fraction expansion has the form

$$X(z) = C + \frac{A_1}{(1 - \frac{1}{2}z^{-1})} + \frac{A_2}{(1 - \frac{1}{4}z^{-1})}$$

The constant C is found by long division:

$$\begin{array}{r|l} & \quad\quad\quad 2 \\ \cline{2-2} \frac{1}{8}z^{-2} - \frac{3}{4}z^{-1} + 1 & \frac{1}{4}z^{-2} - \frac{7}{4}z^{-1} + 4 \\ & \frac{1}{4}z^{-2} - \frac{3}{2}z^{-1} + 2 \\ \cline{2-2} & \quad -\frac{1}{4}z^{-1} + 2 \end{array}$$

Therefore, $C = 2$ and we may write $X(z)$ as follows:

$$X(z) = 2 + \frac{2 - \frac{1}{4}z^{-1}}{(1 - \frac{1}{2}z^{-1})(1 - \frac{1}{4}z^{-1})}$$

Next, for the coefficients A_1 and A_2 we have

$$A_1 = \left[1 - \frac{1}{2}z^{-1}X(z)\right]_{z^{-1}=2} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{(1 - \frac{1}{4}z^{-1})}\right|_{z^{-1}=\mathbf{2}} = \mathbf{3}$$

$$A_2 = \left[1 - \frac{1}{4}z^{-1}X(z)\right]_{z^{-1}=4} = \left.\frac{4 - \frac{7}{4}z^{-1} + \frac{1}{4}z^{-2}}{(1 - \frac{1}{2}z^{-1})}\right|_{z^{-1}=4} = -1$$

Thus, the complete partial fraction expansion becomes

$$X(z) = 2 + \frac{3}{(1 - \frac{1}{2}z^{-1})} - \frac{1}{(1 - \frac{1}{4}z^{-1})}$$

Finally, because the region of convergence is the exterior of the circle $|z| > \frac{1}{2}$, $x(n)$ is the right-sided sequence

$$x(n) = 2\delta(n) + 3\left(\frac{1}{2}\right)^n u(n) - \left(\frac{1}{4}\right)^n u(n)$$

Example 6.6: Find the inverse z-transform of the following $X(z)$.

$$X(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - \frac{3}{2}z^{-1} + \frac{1}{2}z^{-2}} = \frac{(1 + z^{-1})^2}{\left(1 - \frac{1}{2}z^{-1}\right)(1 - z^{-1})},$$

Solution:

$$X(z) = B_0 + \frac{A_1}{1 - \frac{1}{2}z^{-1}} + \frac{A_2}{1 - z^{-1}}.$$

$$\begin{array}{r}
2 \\
\tfrac{1}{2}z^{-2} - \tfrac{3}{2}z^{-1} + 1 \;\big)\; \overline{z^{-2} + 2z^{-1} + 1} \\
\underline{z^{-2} - 3z^{-1} + 2} \\
5z^{-1} - 1
\end{array}$$

$$X(z) = 2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}.$$

$$A_1 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - \frac{1}{2}z^{-1}\right)\right]_{z=1/2} = -9,$$

$$A_2 = \left[\left(2 + \frac{-1 + 5z^{-1}}{\left(1 - \frac{1}{2}z^{-1}\right)\left(1 - z^{-1}\right)}\right)\left(1 - z^{-1}\right)\right]_{z=1} = 8.$$

Therefore,

$$X(z) = 2 - \frac{9}{1 - \frac{1}{2}z^{-1}} + \frac{8}{1 - z^{-1}}.$$

$$x[n] = 2\delta[n] - 9\left(\frac{1}{2}\right)^n u[n] + 8u[n].$$

7. The Discrete Fourier Transform (DFT)

The DFT is an important decomposition for sequences that are finite in length. The DFT is a mapping form a sequence, $x(n)$, to another sequence, $X(k)$,

$$x(n) \overset{DFT}{\Longleftrightarrow} X(k)$$

The sequence $X(k)$ is called the N-point DFT of $x(n)$.

$$\boxed{X(k) = \sum_{n=0}^{N-1} x(n) e^{-j2\pi nk/N} \qquad 0 \le k < N} \tag{7.1}$$

and $x(n)$ may be expanded as follows

$$x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) e^{j2\pi nk/N} \qquad 0 \le n < N \tag{7.2}$$

A notational simplification that is often used for the DFT is to define

$$W_N = e^{-j2\pi/N}$$

For the complex exponential and write the DFT pair as follows:

$$X(k) = \sum_{n=0}^{N-1} x(n) {W_N}^{nk} \qquad 0 \le k < N \tag{7.3}$$

$$\boxed{x(n) = \frac{1}{N} \sum_{k=0}^{N-1} X(k) {W_N}^{-nk} \qquad 0 \le n < N} \tag{7.4}$$

***Example 7.1*:** Find the DFT for the sequence $x = [1\ \ 3\ \ 5\ \ 2]$ **.**

Solution: given that $N = 4$.

$$X(k) = \sum_{n=0}^{N-1} x(n) {W_N}^{nk} \qquad 0 \le k < N$$

$$X(k) = \sum_{n=0}^{3} x(n) {W_4}^{n \times k} \qquad 0 \le k < 4$$

$when\ k = 0$

$$X(0)=\sum_{n=0}^{3}x(n){W_4}^{n\times 0}=1.{W_4}^{0*0}+3.{W_4}^{1*0}+5.{W_4}^{2*0}+2.{W_4}^{3*0}$$

$$=1.{W_4}^{0}+3.{W_4}^{0}+5.{W_4}^{0}+2.{W_4}^{0}$$

$$=1e^{-j\frac{0*2\pi}{4}}+3e^{-j\frac{0*2\pi}{4}}+5e^{-j\frac{0*2\pi}{4}}+2e^{-j\frac{0*2\pi}{4}}$$

$$=1[\cos 0-\mathrm{j}\sin 0]+3[\cos 0-\mathrm{j}\sin 0]+5[\cos 0-\mathrm{j}\sin \pi 0]+2[\cos 0-\mathrm{j}\sin 0]$$

$$=1[1-0]+3[1-0]+5[1-0]+2[1-0]=11$$

$X(0)=11$

$when\ k=1$

$$X(1)=\sum_{n=0}^{3}x(n){W_4}^{n\times 1}=1.{W_4}^{0*1}+3.{W_4}^{1*1}+5.{W_4}^{2*1}+2.{W_4}^{3*1}$$

$$=\ 1.{W_4}^{0}+3.{W_4}^{1}+5.{W_4}^{2}+2.{W_4}^{3}$$

$$=1e^{-j0}+3e^{-j\frac{1*2\pi}{4}}+5e^{-j\frac{2*2\pi}{4}}+2e^{-j\frac{3*2\pi}{4}}$$

$$=1e^{-j0}+3e^{-j\frac{2\pi}{4}}+5e^{-j\frac{4\pi}{4}}+2e^{-j\frac{6\pi}{4}}$$

$$=1e^{-j0}+3e^{-j\frac{\pi}{2}}+5e^{-j\pi}+2e^{-j\frac{3\pi}{2}}$$

$$=1[\cos 0-\mathrm{j}\sin 0]+3\left[\cos\frac{\pi}{2}-\mathrm{j}\sin\frac{\pi}{2}\right]+5[\cos\pi-\mathrm{j}\sin\pi]+2\left[\cos\frac{3\pi}{2}-\mathrm{j}\sin\frac{3\pi}{2}\right]$$

$$=1[1-0]+3[0-j]+5[(-1)-0]+2[0-(-j)]=1-3j-5+2j=-4-j$$

$when\ k=2$

$$X(2)=\sum_{n=0}^{3}x(n){W_4}^{n\times 2}=1.{W_4}^{0*2}+3.{W_4}^{1*2}+5.{W_4}^{2*2}+2.{W_4}^{3*2}$$

$$=1.{W_4}^{0}+3.{W_4}^{2}+5.{W_4}^{4}+2.{W_4}^{6}\quad=1+3$$

$$=1e^{-j0}+3e^{-j\frac{2*2\pi}{4}}+5e^{-j\frac{4*2\pi}{4}}+2e^{-j\frac{6*2\pi}{4}}$$

$$=1e^{-j0}+3e^{-j\frac{4\pi}{4}}+5e^{-j\frac{8\pi}{4}}+2e^{-j\frac{12\pi}{4}}$$

$$=1e^{-j0}+3e^{-j\pi}+5e^{-j2\pi}+2e^{-j3\pi}$$

$$=1[\cos 0-\mathrm{j}\sin 0]+3[\cos\pi-\mathrm{j}\sin\pi]+5[\cos 2\pi-\mathrm{j}\sin 2\pi]+2[\cos 3\pi-\mathrm{j}\sin 3\pi]$$

$$=1[1-0]+3[(-1)-0]+5[1-0]+2[(-1)+0]=1-3+5-2=1$$

$X(2) = 1$

$when\ k = 3$

$$X(3) = \sum_{n=0}^{3} x(n) {W_4}^{n\times 3} = 1.{W_4}^{0*3} + 3.{W_4}^{1*3} + 5.{W_4}^{2*3} + 2.{W_4}^{3*3}$$

$$= 1.{W_4}^{0} + 3.{W_4}^{3} + 5.{W_4}^{6} + 2.{W_4}^{9}$$

$$= 1e^{-j0} + 3e^{-j\frac{3*2\pi}{4}} + 5e^{-j\frac{6*2\pi}{4}} + 2e^{-j\frac{9*2\pi}{4}}$$

$$= 1e^{-j0} + 3e^{-j\frac{6\pi}{4}} + 5e^{-j\frac{12\pi}{4}} + 2e^{-j\frac{18\pi}{4}}$$

$$= 1e^{-j0} + 3e^{-j\frac{3\pi}{2}} + 5e^{-j3\pi} + 2e^{-j\frac{9\pi}{2}}$$

$$= 1[\cos 0 - \mathrm{j}\sin 0] + 3\left[\cos\frac{3\pi}{2} - \mathrm{j}\sin\frac{3\pi}{2}\right] + 5[\cos 3\pi - \mathrm{j}\sin 3\pi] + 2\left[\cos\frac{9\pi}{2} - \mathrm{j}\sin\frac{9\pi}{2}\right]$$

$$= 1[1-0] + 3[0-(-1j)] + 5[(-1)-0] + 2[0-1j] = 1 + 3j - 5 - 2j$$

$$= -4 + j$$

<u>7.1 DFT Properties</u>

- ***Linearity***

If $x_1(n)$ and $x_2(n)$ have N-point DFTs $X_1(k)$ and $X_2(k)$, respectively,

$$a x_1(n) + b x_2(n) \overset{DFT}{\Longleftrightarrow} a X_1(k) + b X_2(k) \tag{7.5}$$

<u>Note</u>: if $x_1(n)$ and $x_2(n)$ have different lengths, the shorter sequence must be added with zeros in order to make it the same length as the longer sequence.

- ***Symmetry***

If $x(n)$ is real-valued, $X(k)$ is conjugate symmetric,

$$X(k) = X^*(-k) = X^*((N-k))_N \tag{7.6}$$

And if $x(n)$ is imaginary, $X(k)$ is conjugate antisymmetrice,

$$X(k) = -X^*(-k) = -X^*((N-k))_N \tag{7.7}$$

Where $((i))_N$ or $(i\ mod\ N)$ are taken to mean "$i\ modulo\ N$" . For example, $((13))_8 = 5$ and $((-6))_8 = 2$.

- ***Circular Shift***

$$x((n - n_o))_N \mathcal{R}_N(n) = \tilde{x}(n - n_o)\mathcal{R}_N(n)$$

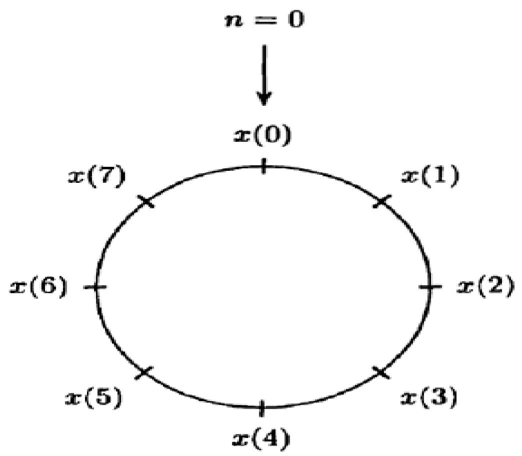
Where n_o is the amount of the shift and $\mathcal{R}_N(n)$ is a rectangular window:

$$\mathcal{R}_N(n) = \begin{cases} 1 & 0 \le n < N \\ 0 & else \end{cases}$$

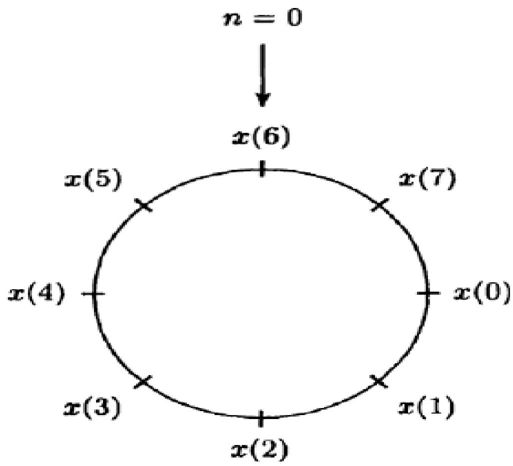
Where $\tilde{x}(n)$ is the periodic sequence which may be formed from $x(n)$ as follows:

$$\tilde{x}(n) = \sum_{k=-\infty}^{\infty} x(n + kN)$$

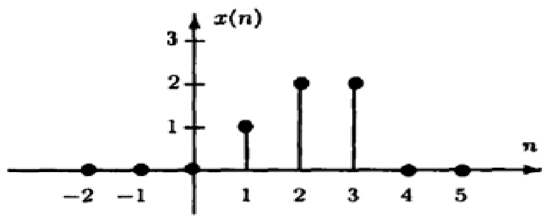
A circular shift to the right by n_o corresponds to a rotation of the circle n_o positions in a clockwise direction.



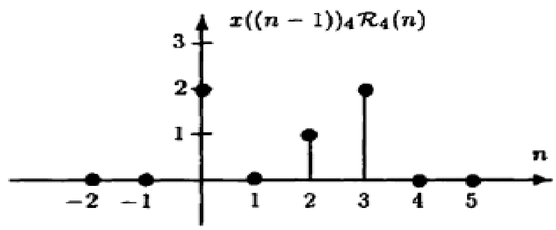
Eight-point sequence.



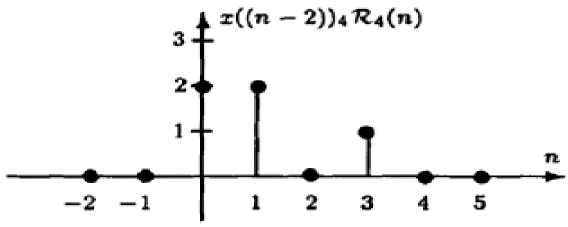
Circular Shift by two



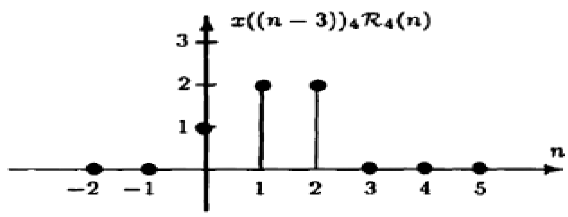
(a) A discrete-time signal of length N=4.



(b) Circular shift by one.



(c) Circular shift by two.



(d) Circular shift by three.

If a sequence is circularly shifted, the DFT is multiplied by a complex exponential,

$$x\big((n-n_o)\big)_N \mathcal{R}_N(n) \overset{DFT}{\Longleftrightarrow} {W_N}^{n_o k} X(k) \tag{7.8}$$

Similarly, with a circular shift of the DFT, $X(k-k_o)$, the sequence is multiplied by a complex exponential,

$$ {W_N}^{nk_o} x(n) \overset{DFT}{\Longleftrightarrow} X((k+k_o))_N \tag{7.9}$$

8. Radix-2Fast Fourier Transform (FFT)

Because $x(n)$ may be either real or complex, evaluating $X(k)$ (see Eqn.(7.3)) requires on the order of N complex multiplications and N complex additions for each value of k.therefore , because there are N value of N, computing an N -point DFT requires N^2 complex multiplications and additions. Suppose that the length of $x(n)$ is even (i.e., N is divisible by 2). If $x(n)$ is decimated into two sequences of length of $N/2$, computing the $N/2$ -point DFT of each of these sequences requires approximately $(N/2)^2$ multiplications and the same number of additions. Thus, the two DFTs require $2(N/2)^2 = \frac{1}{2}N^2$ multiplies and adds. Therefore, if it is possible to find the N-point DFT of $x(n)$ from these two $N/2$-point DFTs in fewer than $\frac{1}{2}N^2$ operations, a savings has been realized.

8.1 Decimation-in-Time FFT

Let $x(n)$ be a sequence of length $N = 2^v$, and suppose that $x(n)$ is split (decimated) into two subsequences, each of length $N/2$. As illustrated in Fig.(8.1), the first sequence, $g(n)$ is formed from the even-index terms,

$$g(n) = x(2n) \qquad\qquad n = 0,1, \dots \frac{N}{2} - 1$$

and the second. $h(n)$, is formed from the odd-index terms,

$$h(n) = x(2n+1) \qquad\qquad n = 0,1, \dots \frac{N}{2} - 1$$

In terms of these sequence, the N-point DFT of $x(n)$ is

$$X(k) = \sum_{n=0}^{N-1} x(n) W_N^{nk} = \sum_{n\ even} x(n) W_N^{nk} + \sum_{n\ odd} x(n) W_N^{nk}$$

$$= \sum_{l=0}^{\frac{N}{2}-1} g(l) W_N^{2lk} + \sum_{l=0}^{\frac{N}{2}-1} h(l) W_N^{(2l+1)k} \quad k = 0,1,\dots,N-1 \tag{8.1}$$

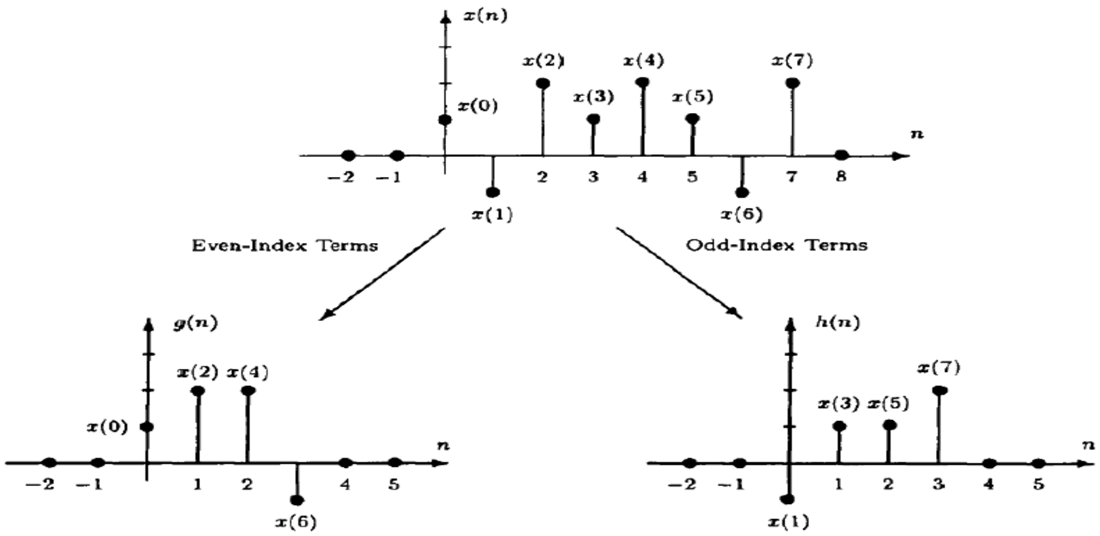


Fig.(8.1)

Because $W_N^{2lk} = W_{N/2}^{lk}$, Eqn.(8.1) may be written as

$$X(k) = \sum_{l=0}^{\frac{N}{2}-1} g(l) W_{N/2}^{lk} + \sum_{l=0}^{\frac{N}{2}-1} h(l) W_{N/2}^{lk} W_N^k \quad k = 0,1,\dots,N-1$$

$$X(k) = \sum_{l=0}^{\frac{N}{2}-1} g(l) W_{N/2}^{lk} + W_N^k \sum_{l=0}^{\frac{N}{2}-1} h(l) W_{N/2}^{lk} \quad k = 0,1,\dots,N-1$$

Note that the first term is the $N/2$ –point DFT of $g(n)$, and the second is the $N/2$ –point DFT of $h(n)$:

$$X(k) = G(k) + W_N^k H(k) \qquad k = 0,1,\dots,N-1 \tag{8.2}$$

Although the $N/2$ -point DFTs of $g(n)$ and $h(n)$ are sequences of length $N/2$, the periodicity of the complex exponentials allows us to write

$$G(k) = G(k + \tfrac{N}{2}) \qquad H(k) = H(k + \tfrac{N}{2})$$

Therefore,$X(k)$ may be computed from the $N/2$ –point DFTs $G(k)$ and $H(k)$. Note that because

$$W_N^{k+N/2} = W_N^k W_N^{N/2} = -W_N^k$$

Then

$$W_N^{k+N/2} H\left(k+\frac{N}{2}\right) = -W_N^k H(k)$$

And it is only necessary to form the products $W_N^k H(k)$ for $k = 0,1,\dots,N/2-1$. A block diagram showing the computations that are necessary for the first stage of an eight-point decimation-in-time FFT is shown in Fig.(8.2).

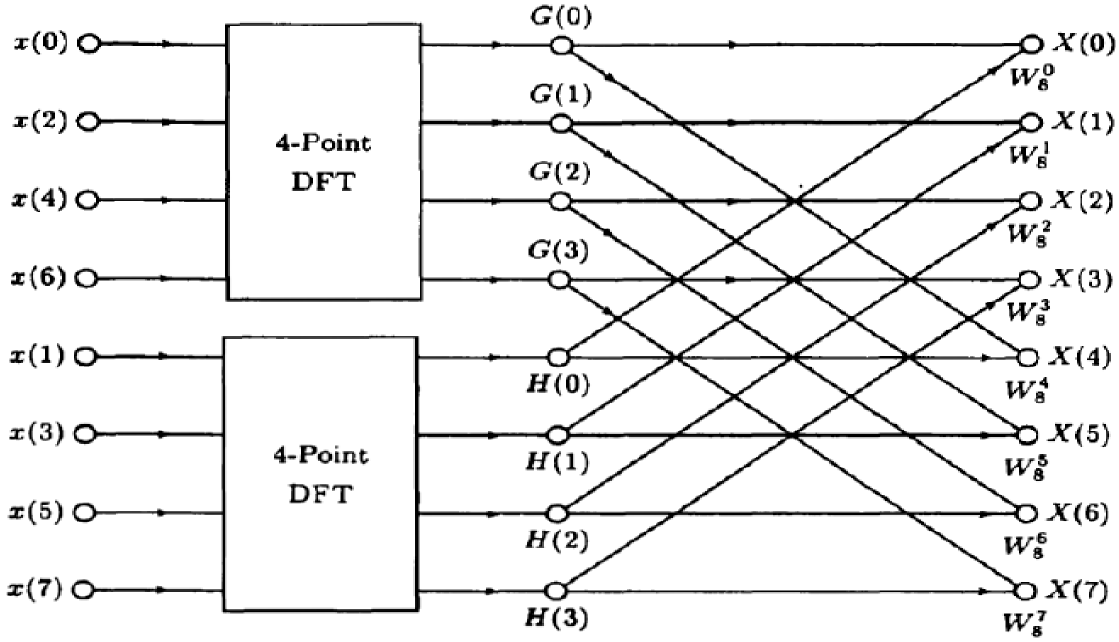


Fig.(8.2)

If $N/2$ is even, $g(n)$ and $h(n)$ may again be decimated. For example, $G(k)$ may be evaluated as follows:

$$G(k) = \sum_{n=0}^{\frac{N}{2}-1} g(n) W_{\frac{N}{2}}^{nk} = \sum_{n\ even}^{\frac{N}{2}-1} g(n) W_{N/2}^{nk} + \sum_{n\ odd}^{\frac{N}{2}-1} g(n) W_{N/2}^{nk}$$

As before, this leads to

$$G(k) = \sum_{n=0}^{\frac{N}{4}-1} g(2n) W_{N/4}^{nk} + W_{N/2}^{nk} \sum_{n=0}^{\frac{N}{4}-1} g(2n+1) W_{N/4}^{nk}$$

Where the first term is the $N/4$ –point DFT of the even samples of $g(n)$ and the second is the $N/4$ –point DFT of the odd samples. A block diagram illustrating this decomposition is shown in Fig.(8.3). If the N is a power of 2, the decimation may be continued until there are only two-point DFTs of the form shown in Fig. (8.4);

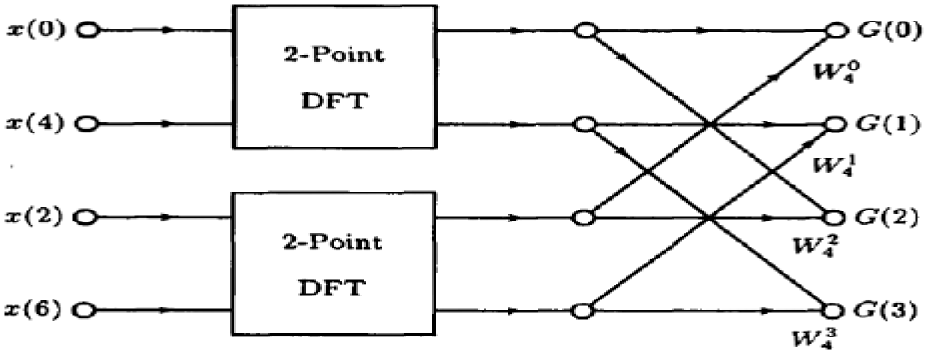


Fig.(8.3)

The basic computational unit of FFT, shown in Fig.(8.4a), is called a butterfly. This structure may be simplified by factoring out a term W_N^r from the lower branch as illustrated in Fig.(8.4b). The factor that remains is $W_N^{N/2} = -1$ . .A complete eight-point radix-2 decimation-in-time FFT is shown in Fig.(8.5).

(a) **(b)**

Fig.(8.4)

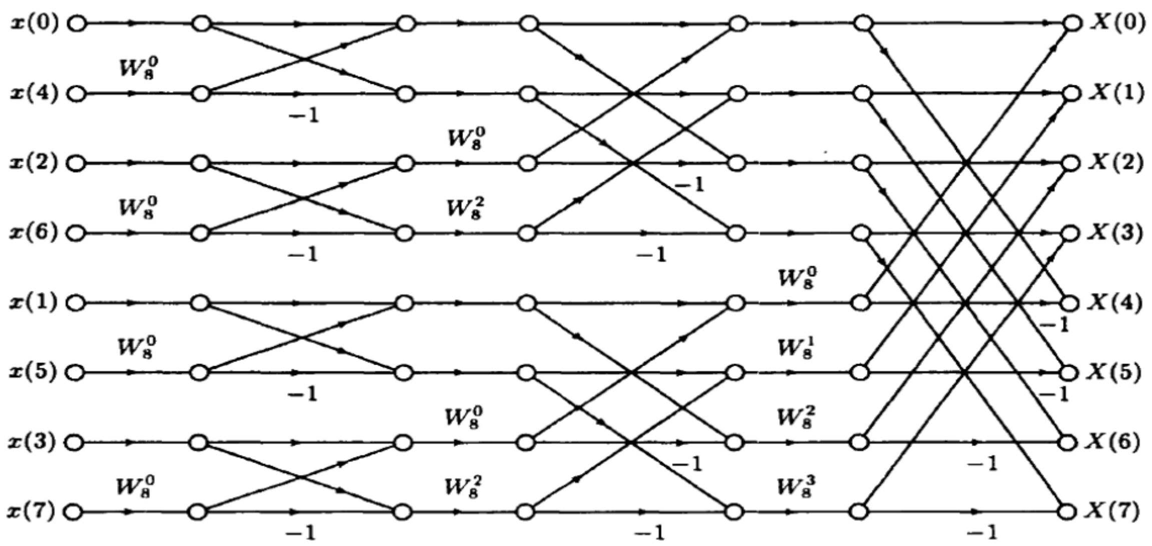


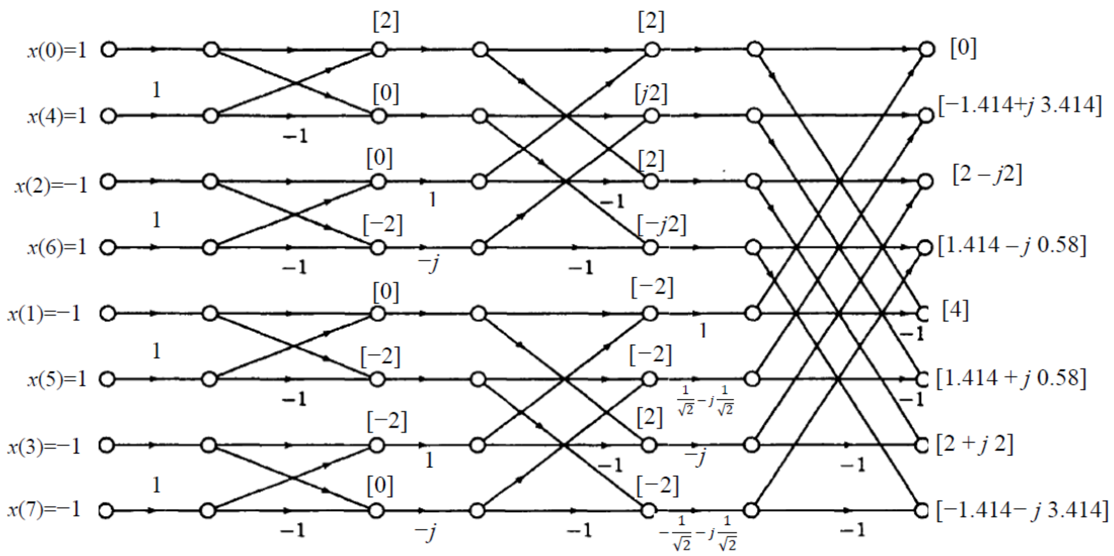
Fig.(8.5): Cooley-Tukey FFT decimation in time.

Example 8.1: Find the DFT of the following sequence x using the FFT algorithm.

$$x = [1, -1, -1, -1, 1, 1, 1, -1]$$

Solution: the scale factors $W_8^k (k = 0, 1, \dots, N/2 - 1)$ are easily calculated as follows:

$$W_8^0 = 1, W_8^1 = e^{-j2\pi/8} = \frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}, W_8^2 = e^{-j4\pi/8} = -j, W_8^3 = e^{-j6\pi/8} = -\frac{1}{\sqrt{2}} - j\frac{1}{\sqrt{2}}$$



Example 8.2: Consider the sequence

$$x(n) = \delta(n) + 2\delta(n-2) + \delta(n-3)$$

(a) Find the four-point ($N = 4$) DFT of $x(n)$.

(b) Confirm your result in (a) using the FFT algorithm

Solution:

(a) $x(n) = \delta(n) + 2\delta(n-2) + \delta(n-3)$

$x(n) = [\ 1\ 0\ 2\ 1]$

$$X(k) = \sum_{n=0}^{N-1} x(n)e^{-j2\pi nk/N} = \sum_{n=0}^{3} x(n)e^{-j2\pi nk/N}$$

$$X(0) = 1 + 0 + 2 + 1 = 4$$

$$X(1) = \sum_{n=0}^{3} x(n)W_4^{\,n\times 1} = 1.W_4^{\,0*1} + 0.W_4^{\,1*1} + 2.W_4^{\,2*1} + 1.W_4^{\,3*1}$$

$$= 1.W_4^{\,0} + 0.W_4^{\,1} + 2.W_4^{\,2} + 1.W_4^{\,3}$$

$$X(1) = 1e^{-j0} + 0e^{-j\frac{1*2\pi}{4}} + 2e^{-j\frac{2*2\pi}{4}} + 1e^{-j\frac{3*2\pi}{4}}$$

$$= 1e^{-j0} + 0e^{-j\frac{2\pi}{4}} + 2e^{-j\frac{4\pi}{4}} + 1e^{-j\frac{6\pi}{4}}$$

$$= 1e^{-j0} + 0e^{-j\frac{\pi}{2}} + 2e^{-j\pi} + 1e^{-j\frac{3\pi}{2}}$$

$$= 1[\cos 0 - \text{j}\sin 0] + 0\left[\cos\frac{\pi}{2} - \text{j}\sin\frac{\pi}{2}\right] + 2[\cos\pi - \text{j}\sin\pi] + 1\left[\cos\frac{3\pi}{2} - \text{j}\sin\frac{3\pi}{2}\right]$$

$$= 1[1-0] + 0[0-j] + 2[(-1)-0] + 1[0-(-j)] = 1 - 2 + j = -1 + j$$

$$X(2) = \sum_{n=0}^{3} x(n)W_4^{\,n\times 2} = 1.W_4^{\,0*2} + 0.W_4^{\,1*2} + 2.W_4^{\,2*2} + 1.W_4^{\,3*2}$$

$$= 1.W_4^{\,0} + 0.W_4^{\,2} + 2.W_4^{\,4} + 1.W_4^{\,6} \quad = 1 + 3$$

$$= 1e^{-j0} + 0e^{-j\frac{2*2\pi}{4}} + 2e^{-j\frac{4*2\pi}{4}} + 1e^{-j\frac{6*2\pi}{4}}$$

$$= 1e^{-j0} + 0e^{-j\frac{4\pi}{4}} + 2e^{-j\frac{8\pi}{4}} + 1e^{-j\frac{12\pi}{4}}$$

$$= 1e^{-j0} + 0e^{-j\pi} + 2e^{-j2\pi} + 1e^{-j3\pi}$$

$$= 1[\cos 0 - \text{j}\sin 0] + 0[\cos\pi - \text{j}\sin\pi] + 2[\cos 2\pi - \text{j}\sin 2\pi] + 1[\cos 3\pi - \text{j}\sin 3\pi]$$

$$= 1[1 - 0] + 0[(-1) - 0] + 2[1 - 0] + 1[(-1) + 0] = 1 + 0 + 2 - 1 = 2$$

$$X(3) = \sum_{n=0}^{3} x(n){W_4}^{n\times 3} = 1.{W_4}^{0*3} + 0.{W_4}^{1*3} + 2.{W_4}^{2*3} + 1.{W_4}^{3*3}$$

$$= 1.{W_4}^{0} + 0.{W_4}^{3} + 2.{W_4}^{6} + 1.{W_4}^{9} \quad =$$

$$= 1e^{-j0} + 0e^{-j\frac{3*2\pi}{4}} + 2e^{-j\frac{6*2\pi}{4}} + 1e^{-j\frac{9*2\pi}{4}}$$

$$= 1e^{-j0} + 0e^{-j\frac{6\pi}{4}} + 2e^{-j\frac{12\pi}{4}} + 1e^{-j\frac{18\pi}{4}}$$

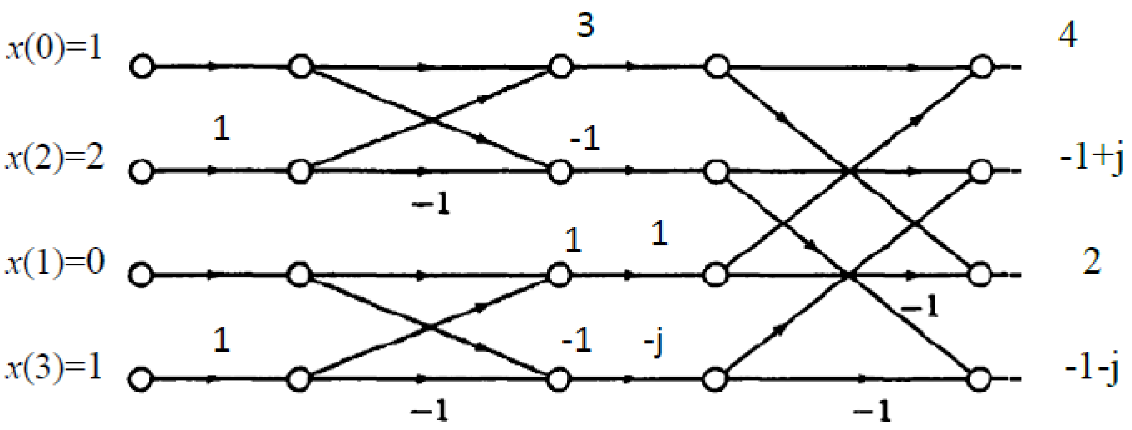
$$= 1e^{-j0} + 0e^{-j\frac{3\pi}{2}} + 2e^{-j3\pi} + 1e^{-j\frac{9\pi}{2}}$$

$$= 1[\cos 0 - \mathrm{j}\sin 0] + 0\left[\cos\frac{3\pi}{2} - \mathrm{j}\sin\frac{3\pi}{2}\right] + 2[\cos 3\pi - \mathrm{j}\sin 3\pi]$$
$$+ 1\left[\cos\frac{9\pi}{2} - \mathrm{j}\sin\frac{9\pi}{2}\right]$$

$$= 1[1 - 0] + 0[0 - (-1j)] + 2[(-1) - 0] + 1[0 - 1j] = 1 + 0 - 2 - j = -1 - j$$

$$X(k) = [4 \quad -1 + j1 \quad 2 \quad -1 - j1]$$

(b) $\qquad W_4^0 = 1, W_4^1 = -j$



8.2 Complexity of FFT

Computing an N-point DFT using a radix-2 decimation-in-time FFT is much more efficient than calculation the DFT directly. For example, if $N = 2^v$, there are $\log_2 N = v$ stages of computation. Because each stage requires $N/2$ complex multiplies by the factors W_N^r and N complex additions, there are a total of $\frac{1}{2} N \log_2 N$ complex multiplications and $N \log_2 N$ complex additions.

8.3 Inverse Fast Fourier-Transform(IFFT)

It is possible to calculate the IFFT using FFT algorithm:

$$X(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N^{-nk} = \frac{1}{N} j \sum_{k=0}^{N-1} -jX(k) W_N^{-nk} = \frac{1}{N} j \left[\sum_{k=0}^{N-1} jX^*(k) W_N^{nk}\right]^*$$

$$= \frac{1}{N} j [FFT(j\, X^*(k))]^*$$

The algorithm can be summarized by the following steps:

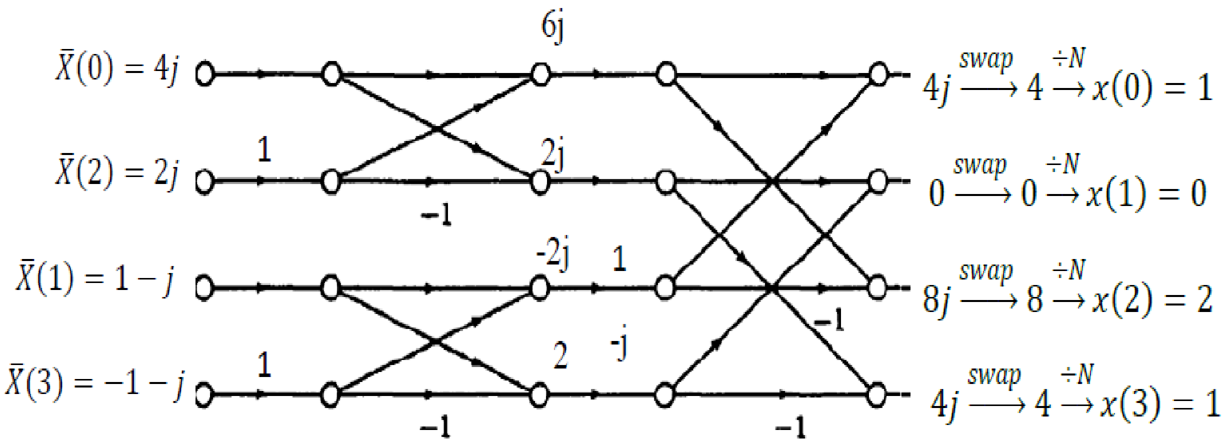
1. FFT of sequence $j\, X^*(k)$, i.e. swap real and imaginary parts.
2. Swap real and imaginary parts of result.
3. Normalize $1/N$.

Example 8.3: Find the IFFT the sequence resulted from example 8.2.

$$X(k) = [4, -1 + j, 2, -1 - j]$$

Solution:

$$\bar{X}(k) = jX^*(k) = [4j, 1 - j, 2j, -1 - j]$$



9.Convolution

The relationship between the input to a linear shift-invariant system, $x(n)$, and the output, $y(n)$, is given by the convolution sum

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) \qquad (9.1)$$

9.1 Convolution Properties

- ***Commutative Property***

$$x(n) * h(n) = h(n) * x(n)$$

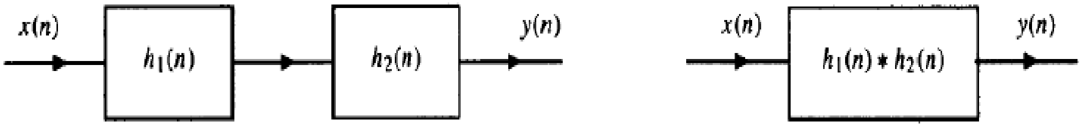
- ***Associative Property***

$$\{x(n) * h_1(n)\} * h_2(\mathrm{n}) = x(n) * \{h_1(n) * h_2(\mathrm{n})\}$$

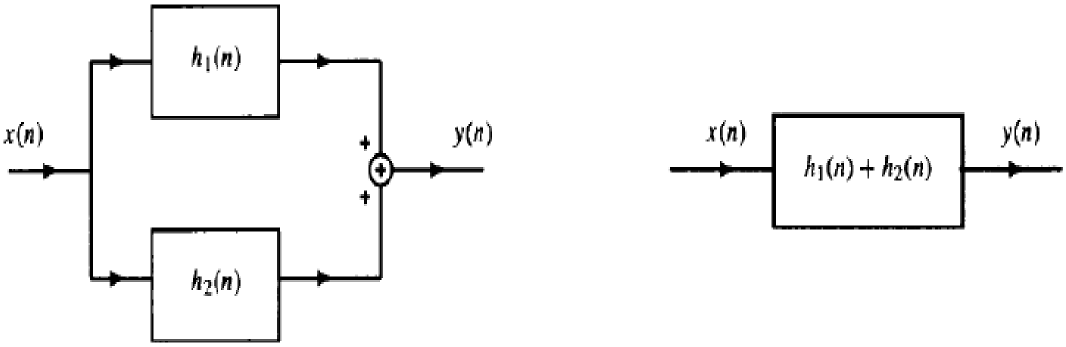
- ***Distributive Property***

$$x(n) * \{h_1(n) + h_2(n)\} = x(n) * h_1(n) + x(n) * h_2(n)$$

(a) The commutative property.



(b) The associative property.



(c) The distributive property.

9.2 Performing Convolution

1. Direct Evaluation

$\sum_{n=0}^{N-1} a^n = \frac{1-a^N}{1-a}$	$\sum_{n=0}^{\infty} a^n = \frac{1}{1-a} \qquad \lvert a\rvert < 1$
$\sum_{n=0}^{N-1} na^n = \frac{(N-1)a^{N+1} - Na^N + a}{(1-a)^2}$	$\sum_{n=0}^{\infty} na^n = \frac{a}{(1-a)^2} \qquad \lvert a\rvert < 1$
$\sum_{n=0}^{N-1} n = \frac{1}{2}N(N-1)$	$\sum_{n=0}^{N-1} n^2 = \frac{1}{6}N(N-1)(2N-1)$

Example 9.1 :- Let us perform the convolution of the two signals

$$x(n) = a^n u(n) = \begin{cases} a^n & n \geq 0 \\ 0 & n < 0 \end{cases} \qquad \text{and} \qquad h(n) = u(n)$$

$$y(n) = x(n) * h(n) = \sum_{k=-\infty}^{\infty} x(k)h(n-k) = \sum_{k=-\infty}^{\infty} a^n u(k)u(n-k)$$

Because $u(k)$is equal to zero for $k < 0$ and $u(n-k)$is equal to zero for $k > n$, when $n < 0$, there are no nonzero terms in the sum and $y(n) = 0$. On the other hand, if $n \geq 0$,

$$y(n) = \sum_{k=0}^{n} a^k = \frac{1 - a^{n+1}}{1 - a}$$

Therefore form table above , $y(n) = \sum_{k=0}^{n} a^k = \frac{1-a^{n+1}}{1-a} u(n)$

<u>2. Graphical Approach</u>

Convolutions may also be performed graphically. The steps involved in using the graphical approach are as follows:

1. Plot both sequences, $x(k)$ and $h(k)$,as functions of k.
2. Choose one of the sequences, say $h(k)$, and time-reverse it to form the sequence $h(-k)$.
3. Shift the time-reversed sequences by n.
4. Multiply the two sequences $x(k)$ and $h(n-k)$ and sum the product for all values of k. The resulting value will be equal to $y(n)$. This process is repeated for all possible shifts, n .

   If $x(n)$ is of length L_1 and $h(n)$ is of length L_2 , $y(n) = x(n) * h(n)$ will be of length

$$L = L_1 + L_2 - 1$$

Furthermore, if

$$\{x(n); n_{xb} \leq n \leq n_{xe}\} \text{ and } \{h(n);\ n_{hb} \leq n \leq n_{he}\} \text{ , then}$$

$$\{y(n); n_{yb} \leq n \leq n_{ye}\} \text{ where } n_{yb} = n_{xb} + n_{hb} \text{ and } n_{ye} = n_{xe} + n_{he}$$

Example 9.2 : To illustrate the graphical approach to convolution, let us evaluate $y(n) = x(n) * h(n)$ Where $x(n)$ and $h(n)$ are the sequences shown in Fig.9-1 (a) and (b) , respectively. To perform this convolution, we follow the steps listed above:

1. Because $x(k)$ and $h(k)$ are both plotted as a function of k in Fig.9-1 (a) and (b), we next choose one of the sequences to reverse in time. In this example, we time – reverse $h(k)$, which is shown in Fig.9-1 (c).
2. Forming the product, $x(k)h(-k)$, and summing over k , we find that $y(0) = 1$.
3. Shift $h(k)$ to the right by one results in the sequence $h(1-k)$ shown in Fig.9-1(d). Forming the product, $x(k)h(1-k)$, and summing over k , we find that $y(1) = 3$.
4. Shift $h(1-k)$ to the right again gives the sequence $h(2-k)$ shown in Fig.9-1(e). Forming the product, $x(k)h(2-k)$, and summing over k , we find that $y(2) = 6$.
5. Continuing in this manner, we find that $y(3) = 5$, $y(4) = 3$, and $y(n) = 0$ for $n > 4$.
6. We next take $h(-k)$ and shift it to the left by one as shown in Fig. 9-1(f). Because the product, $x(k)h(-1-k)$, is equal to zero for all k, we find that $y(-1) = 0$. In fact $y(n) = 0$ for all $n < 0$. figure 9-1(g) shows the convolution for all n.

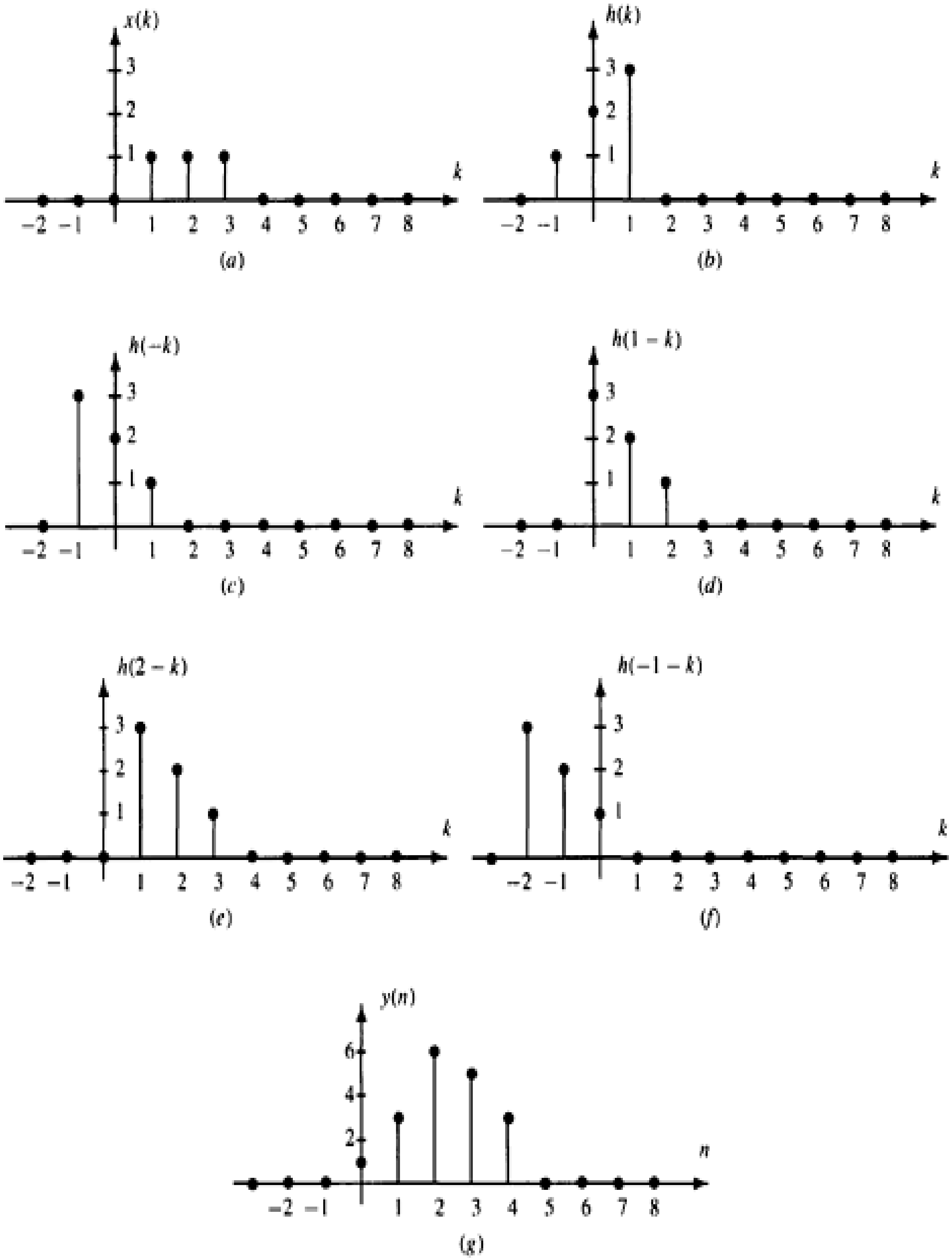


Fig.(9.1). The graphical approach to convolution.

3. Linear Convolution Using The DFT

The DFT provides a convenient way to perform convolutions with having to evaluate the convolution sum. Specifically, if $h(n)$ is N_1 points long and $x(n)$ is N_2 points long. $h(n)$ may be linearly convolved with $x(n)$ as follows:

1. Pad the sequences $h(n)$ and $x(n)$ with zeros so that they are of length $N \geq N_1 + N_2 - 1$.
2. Find the N-point DFTs(use FFT to reduce complexity) of $h(n)$ and $x(n)$.
3. Multiply the DFTs to form the product $Y(k) = H(k)X(k)$.
4. Find the inverse DFT of $Y(k)$.

Example 9.3 : let us consider the sequences $x(n) = [\bar{1}, 2,3,1]$; $h(n) = [\bar{4}, 3,2,1]$ (note: the dash over the number refers to the index $n = 0$) .

Solution: $N_1 = N_2 = 4$,hance $N \geq 4 + 4 - 1 = 7$. To utilize the benefit of FFT, let $N = 8$, and $x_n = [\bar{1}, 2,3,1,0,0,0,0]$ and $h_n = [\bar{4}, 3,2,1,0,0,0,0]$.

Then

$$X(k) = [7,1.7071 - 5.1213j, -2 - 1j, 0.2929 + 0.8787j, 1,0.2929 - 0.8787j, -2 + 1j, 1.7071 + 5.1213j]$$

and

$$H(k) = [10,5.4142 - 4.8284j, 2 - 2j, 2.5858 - 0.8284j, 2,2.5858 + 0.8284j, 2 + 2j, 5.4142 + 4.8284j]$$

$$Y(k) = H(k)X(k) = [70, -15.4853 - 35.9706j, -6 + 2j, 1.4853 + 2.0294j, 2,1.4853 - 2.0294j, -6 - 2j, -15.4853 + 35.9706j]$$

Applying inverse DFT to $Y(k)$ gives

$$x(n) = \frac{1}{N}\sum_{k=0}^{N-1} X(k) W_N{}^{-nk} \qquad 0 \leq n < N$$

$$y_n = [4 ,11,20,18,11,5,1,0]$$

Or

$$y(n) = 4\delta(n) + 11\delta(n-1) + 20\delta(n-2) + 18\delta(n-3) + 11\delta(n-4) + 5\delta(n-5) + \delta(n-6)$$

CHAPTER TWO:DIGITAL FILTER DESIGN

1.Structures for IIR Systems

1.1 Direct Form I

The input $x(n)$ and output $y(n)$ of a causal (Infinite Impulse Response) IIR filter with a rational system function

$$H(z) = \frac{Y(z)}{X(z)} = \frac{\sum_{k=0}^{M} b_k z^{-k}}{1 + \sum_{k=1}^{N} a_k z^{-k}} \qquad (1.1)$$

Is described by the linear constant coefficient difference equation

$$y(n) + \sum_{k=1}^{N} a_k\, y(n-k) = \sum_{k=0}^{M} b_k\, x(n-k) \qquad (1.2)$$

or,

$$y(n) = \sum_{k=0}^{M} b_k\, x(n-k) - \sum_{k=1}^{N} a_k\, y(n-k) \qquad (1.3)$$

The block diagram of Fig.(1.1) is an explicit pictorial representation of Eq.(1.3). More precisely, it represent the pair of difference equations

$$v(n) = \sum_{k=0}^{M} b_k\, x(n-k) \qquad (1.4a)$$

$$y(n) = v(n) - \sum_{k=1}^{N} a_k\, y(n-k) \qquad (1.4b)$$

Form Eqn.(1,1), Fig.(1.1) can be viewed as an implementation of $H(z)$ through the decomposition

$$H(z) = H_2(z)H_1(z) = \left(\frac{1}{1 + \sum_{k=1}^{N} a_k\, z^{-k}}\right)\left(\sum_{k=0}^{M} b_k\, z^{-k}\right) \qquad (1.5)$$

Or, equivalently, through the pair of equations

$$V(z) = H_1(z)X(z) = \left(\sum_{k=0}^{M} b_k\, z^{-k}\right)X(z) \qquad (1.6a)$$

$$Y(z) = H_2(z)V(z) = \left(\frac{1}{1+\sum_{k=1}^{N} a_k\, z^{-k}}\right)V(z) \qquad (1.6b)$$

Figure (1.1) can be viewed as a cascade of two systems, the first representing the computation of v(n) from x(n) and the second representing the computation of y(n) from v(n) .

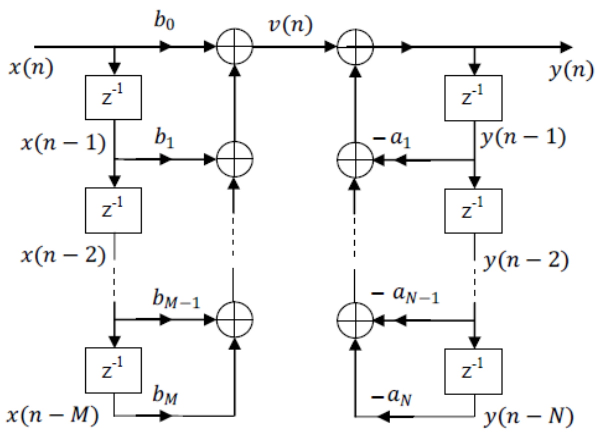


Fig.(1.1)

1.2 Direct Form II

Since each of the two systems is a linear time-invariant system, the order in which the two systems are cascaded can be reversed, as shown in Fig.(1.2), without affecting the overall system function. For convenience, we have assumed that $M = N$.

$$H(z) = H_1(z)H_2(z) = \left(\sum_{k=0}^{M} b_k\, z^{-k}\right)\left(\frac{1}{1+\sum_{k=1}^{N} a_k\, z^{-k}}\right) \qquad (1.7)$$

Or, equivalently, through the pair of equations

$$W(z) = H_2(z)X(z) = \left(\frac{1}{1+\sum_{k=1}^{N} a_k\, z^{-k}}\right)X(z) \qquad (1.8a)$$

$$Y(z) = H_1(z)W(z) = \left(\sum_{k=0}^{M} b_k\, z^{-k}\right)W(z) \qquad (1.8b)$$

In the time domain, Fig.(1.1) and (1.2), equivalently, Eqn. (1.8) can be by the pair of difference equations

$$w(n) = x(n) - \sum_{k=1}^{N} a_k\, w(n-k) \qquad (1.9a)$$

$$y(n) = \sum_{k=0}^{M} b_k\, w(n-k) \qquad (1.9b)$$

The systems in Fig.(1.1) and (1.2) each have a total of $N + M$ delay elements. However, the block diagram of Fig.(1.2) can be redrawn by noting that exactly the same single, $w(n)$, is stored in the two chains of delay elements in the figure. Consequently, the two can be collapsed into one chain, as indicated in Fig.(1.3).
The total number of delay elements in Fig.(1.3) is less than in either Fig.(1.1) or Fig.(1.2). Specifically, the minimum number of delays required is, in general, max (N, M).

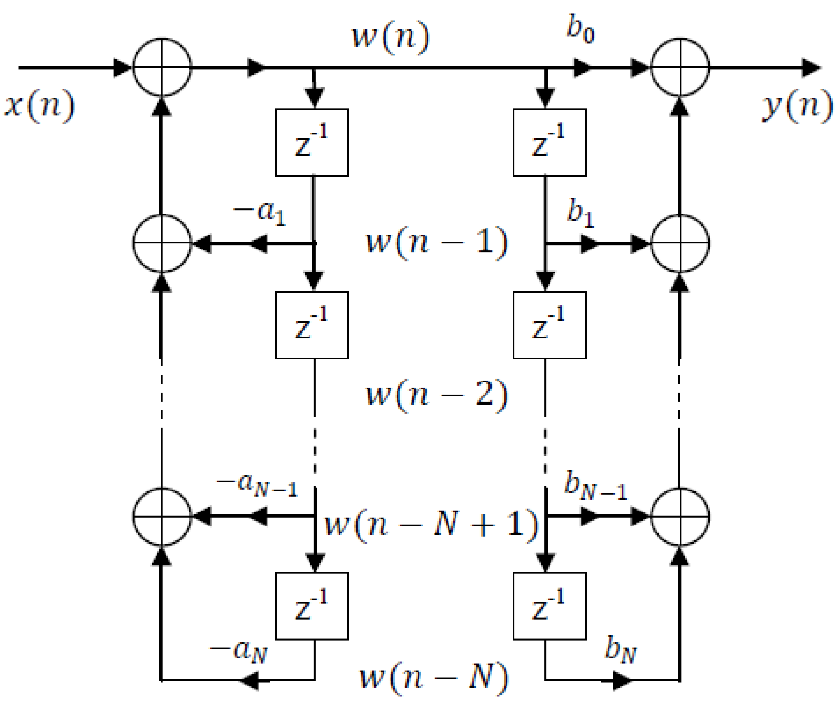


Fig.(1.2)

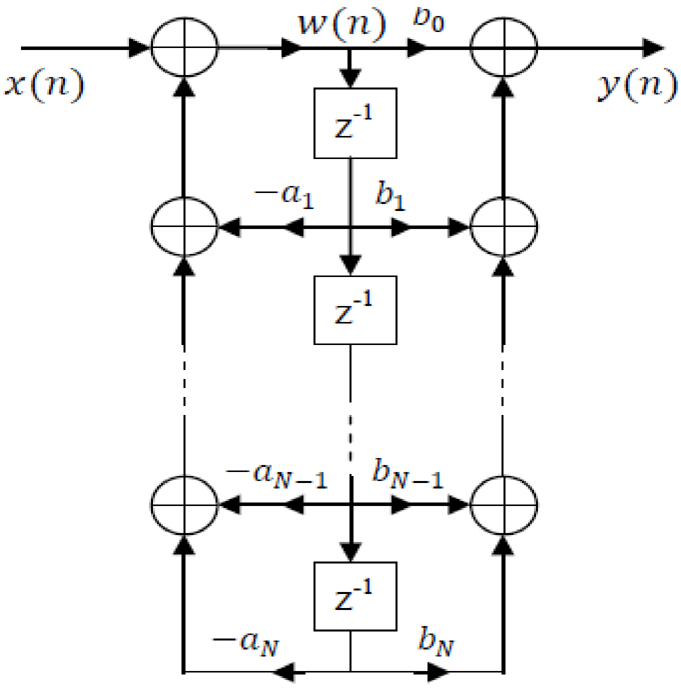


Fig.(1.3)

Example 1.1: Draw the block diagram and the signal flow graph using direct form I and II realization of the discrete-time system represented by the transfer function

$$H(z) = \frac{1 + 2z^{-1}}{1 - 1.5z^{-1} + 0.9z^{-2}}$$

Solution: Comparing this system function with Eqn.(1.1), we find $b_o = 1$, $b_1 = 2$, $a_1 = -1.5$, and $a_2 = 0.9$. Figure (1.4a) and (1.4b) depict a pictorial diagram for the direct form I and II, respectively. Figure (1.4a) and (1.4b) can be rewritten applying the signal flow graph, as shown in Fig.(1.5a) and (1.5b)

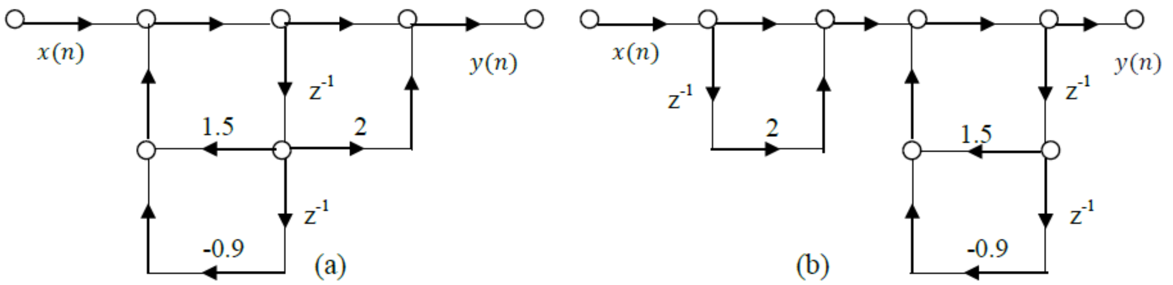


Fig.(1.4)

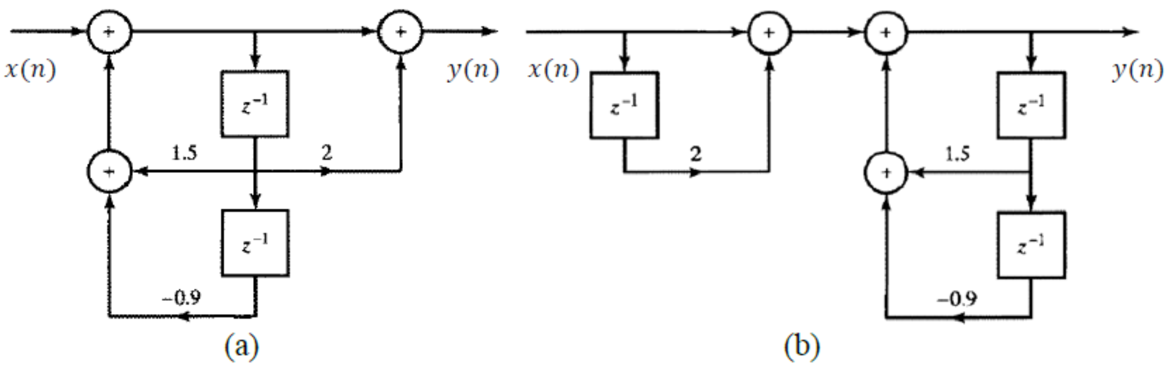


Fig.(1.5)

1.3 Cascade Structure

The cascade structure is derived by factoring the numerator and denominator polynomials of $H(z)$

$$H(z) = \frac{\sum_{k=0}^{N} b_k\, z^{-k}}{1 + \sum_{k=1}^{M} a_k\, z^{-k}} = A \prod_{k=1}^{\max(N,M)} \frac{1 - \beta_k\, z^{-k}}{1 - \alpha_k\, z^{-k}} \qquad (1.10)$$

This factorization corresponds to a cascade of first-order filters, each having one pole and one zero. In general the coefficients α_k and β_k will be complex. However, if $h(n)$ is real, the roots of $H(z)$ will occur in complex conjugate pairs, and these complex conjugate factors may be combined to form second-order factors with real coefficients:

$$H_k(z) = \frac{1 + \beta_{1k}\, z^{-1} + \beta_{2k}\, z^{-2}}{1+\alpha_{1k}\, z^{-1} + \alpha_{2k}\, z^{-2}}$$

There is considerable flexibility in how a system may be implemented in cascade form. For example, there are different pairings of the poles and zeros and different ways in which the sections may be ordered. For example the system

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125\, z^{-2}}$$

Has a direct form I and direct form II structures shown in Fig.(1.6)

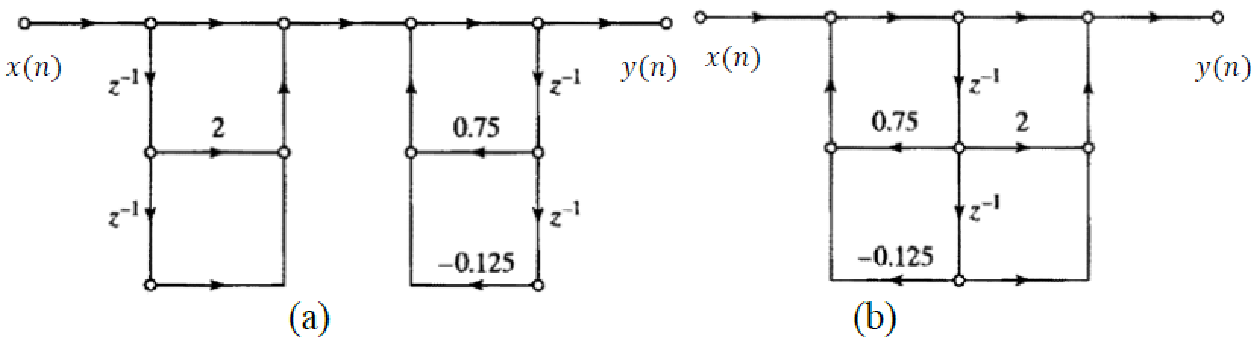


Fig.(1.6)

Alternatively, to illustrate the cascade structure, we can use first-order systems by expressing $H(z)$ as a product of first-order factors, as in

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125\,z^{-2}}$$

$$H(z) = \frac{(1 + z^{-1})(1 + z^{-1})}{(1 - 0.5z^{-1})(1 + 0.25\,z^{-1})}$$

Since all of the poles and zeros are real, a cascade structure with first-order sections has real coefficients. If the poles and/or zeros were complex, only a second-order section would have real coefficients. Fig.(1.7) show two equivalent cascade structures.

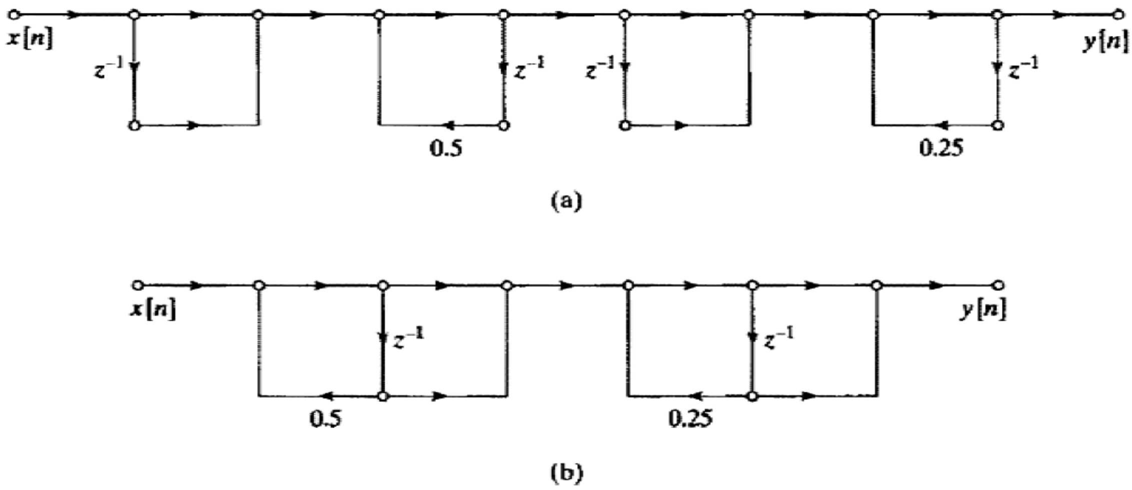


Fig. (1.7): (a) Direct form I subsections. (b) Direct form II subsections.

1.4 Parallel Structure

An alternative to factoring $H(z)$ is to expand the system function using a partial fraction expansion. For example, with

$$H(z) = A \frac{\prod_{k=1}^{M}(1 - \beta_k \, z^{-1})}{\prod_{k=1}^{N}(1 - \alpha_k \, z^{-1})}$$

If $N > M$ and $\alpha_i \neq \alpha_k$ (the roots of the denominator polynomial are distinct), $H(z)$ may be expanded as a sum of N first-order factors as follows:

$$H(z) = \sum_{k=1}^{N} \frac{A_k}{1 - \alpha_k \, z^{-1}}$$

Where the coefficient A_k and α_k are, in general, complex. This expansion corresponds to a sum of N first-order system functions and may be realized by connecting these system in parallel. If $h(n)$ is real, the poles $H(z)$ of will occur in complex conjugate pairs, and these complex roots in the partial fraction expansion may be combined to form second-order systems with real coefficients:

$$H(z) = \sum_{k=1}^{N_s} \frac{\gamma_{0k+}\gamma_{1k} \, z^{-1}}{1 + \alpha_{1k} \, z^{-1} + \alpha_{2k} \, z^{-2}}$$

Shown in Fig.(1.8) is a sixth-order filter implemented as a parallel connection of three second-order direct form II systems. If N $\leq M$, the partial fraction expansion will also contain a term of the form

$$c_0 + c_1 z^{-1} + \cdots + c_{M-N} z^{-(M-N)}$$

Which is an FIR filter that is placed in parallel with the other terms in the expansion of $H(z)$.

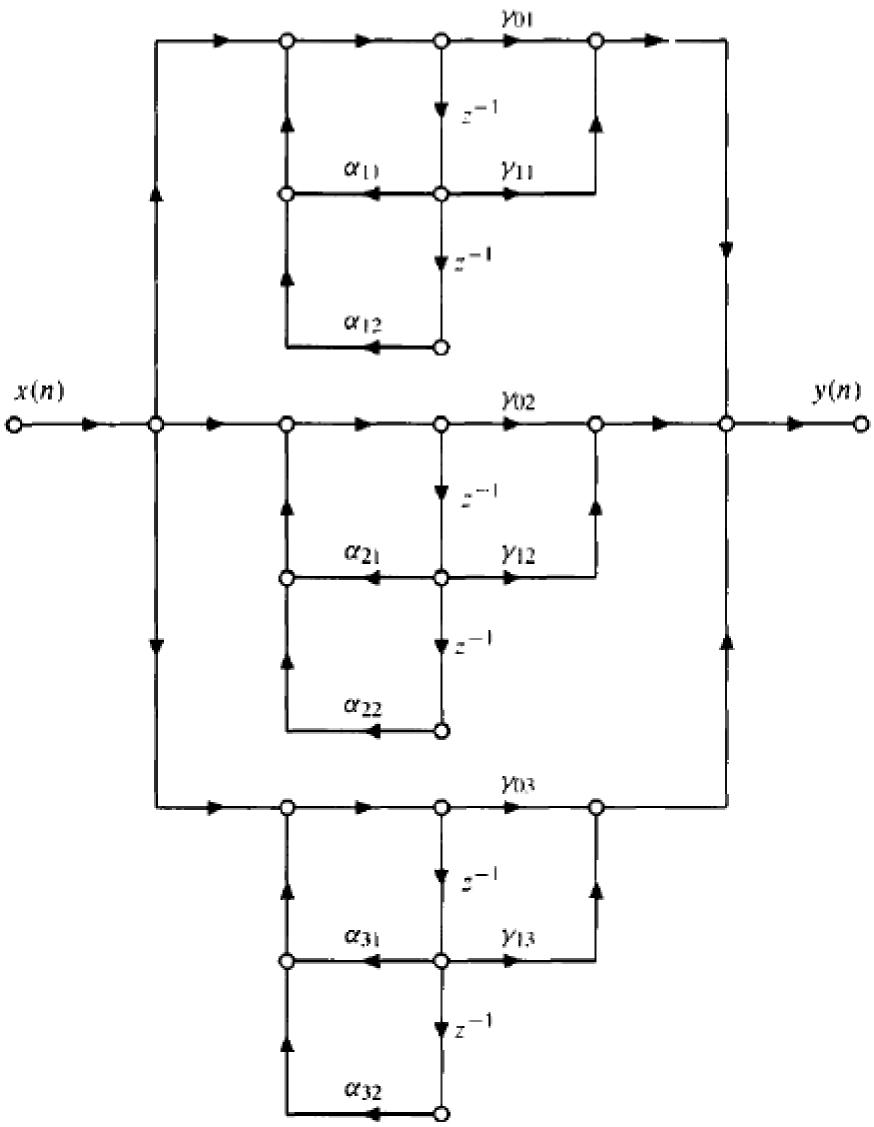


Fig.(1.8)

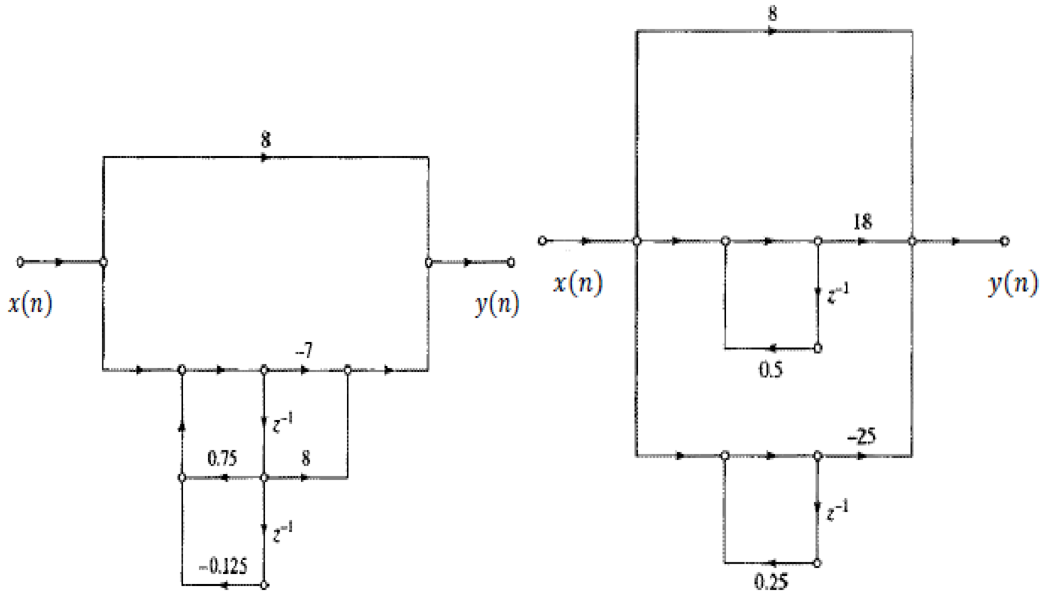
The parallel-form realization for the system with a second-order section is shown in Fig.(1.9a).

$$H(z) = \frac{1 + 2z^{-1} + z^{-2}}{1 - 0.75z^{-1} + 0.125\, z^{-2}} = 8 + \frac{-7 + 8z^{-1}}{1 - 0.75z^{-1} + 0.125\, z^{-2}}$$

Since all the poles are real, we can obtain an alternative parallel form realization by expanding $H(z)$ as

$$H(z) = 8 + \frac{18}{1 - 0.5z^{-1}} - \frac{25}{1 - 0.25z^{-1}}$$

The resulting parallel form with first-order sections is shown in Fig.(1.9b).



(a) (b)

Fig.(1.9)

2.Structures for FIR Systems

A causal FIR filter has a system function that is a polynomial in z^{-1} :

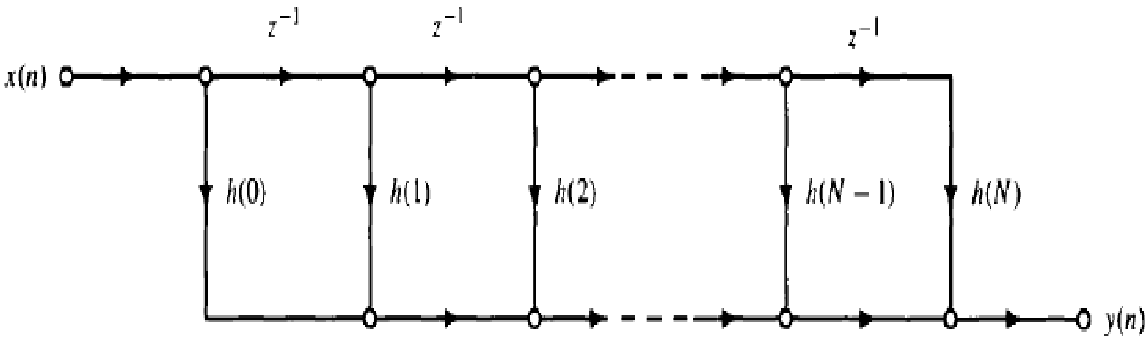
$$H(z) = \sum_{n=0}^{N} h(n)\, z^{-1}$$

For an input $x(n)$, the output is

$$H(z) = \sum_{k=0}^{N} h(k)\, x(n-k)$$

2.1 Direct Form

The most common way to implement an FIR filter is in direct from using a tapped delay line as shown in the figure below



2.2 Cascade Form

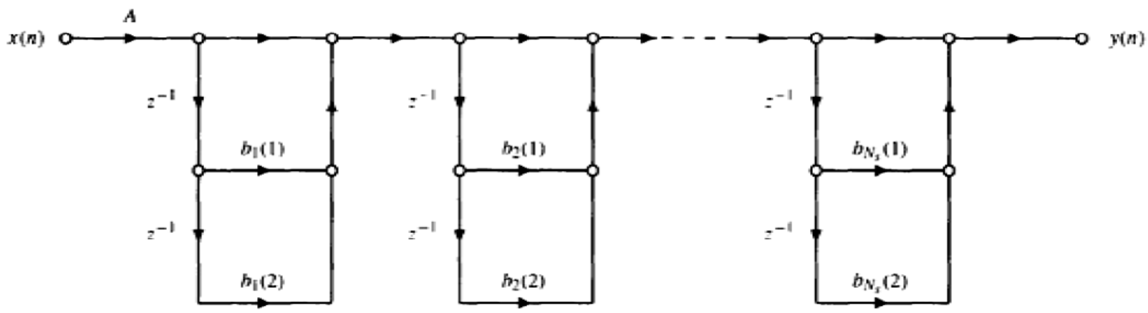
For a causal FIR filter, the system function may be factored into a product of first-order factors,

$$H(z) = \sum_{n=0}^{N} h(n)\, z^{-n} = A \prod_{k=1}^{N} (1 - \alpha_k\, z^{-1})$$

Where α_k for $k = 1, \dots, N$ are the zeros of $H(z)$. If $h(n)$ is real, the complex roots of $H(z)$ occur in complex conjugate pairs, and these conjugate pairs may be combined to form second-order factors with real coefficients,

$$H(z) = A \prod_{k=1}^{N_S} [1 + b_k\,(1) z^{-1} + b_k\,(2) z^{-2}]$$

$H(z)$ may be implemented as a cascade of second-order FIR filter as illustrated in Figure below.



3.IIR FILTER DESIGN

3.1 Butterworth Filters

A unity-gain Butterworth low-pass filter has a transfer function whose magnitude is given by

$$|H_n(j\Omega)| = \frac{1}{\sqrt{1+\left(\frac{\Omega}{\Omega_c}\right)^{2n}}} \tag{3.1}$$

Where n is an integer that denoted the order of the filter.

1. The cutoff frequency is Ω_c rad/s for all value of n.
2. If n is large enough, the denominator is always close to unity when $\Omega < \Omega_c$
3. In the expression for $|H_n(j\Omega)|$, the exponent of is always Ω/Ω_c even.

To derive $H(s)$, let us set $\Omega_c = 1\ rad/s$ (prototype filter), and note that

$$|H_n(j\Omega)|^2 = H_n(j\Omega)H_n(-j\Omega) = \frac{1}{1+\Omega^{2n}}$$

But because $s = j\Omega$, we can write

$$|H_n(s)|^2 = H_n(s)H_n(-s)$$

Thus,

$$|H_n(s)|^2 = \frac{1}{1+(s/j)^{2n}}$$

The procedure for finding $H_n(s)$ for a given value of n is as follows:

1. Find the roots of the polynomial

$$1+(s/j)^{2n} = 0$$

or

$$s^{2n} = -1(j)^{2n} = (-1)^{n+1}$$

for n odd: $s_k = 1\angle k\pi/n,$ $k = 0, 1, 2, \ldots, 2n-1$

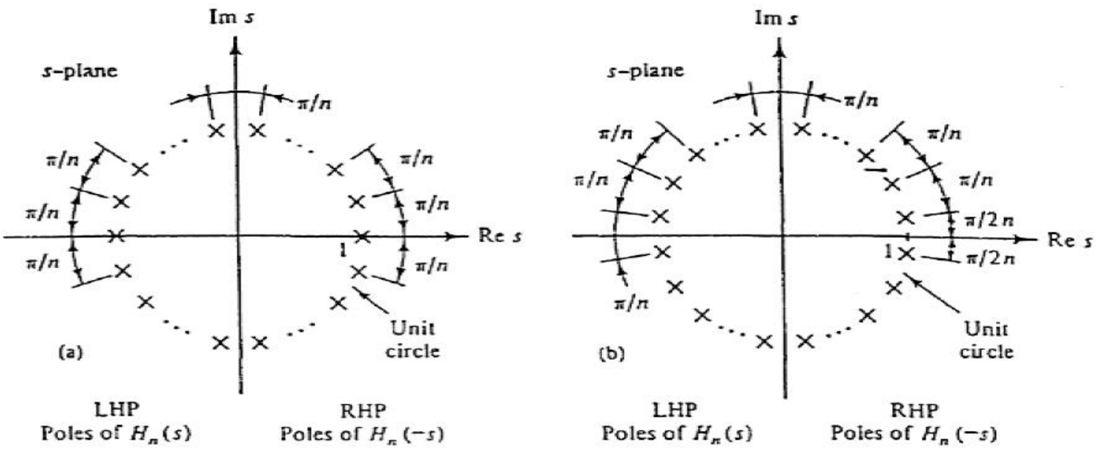
for n even: $s_k = 1\angle \pi/2n + k\pi/n,$ $k = 0, 1, 2, \ldots, 2n-1$

2. Assign the left-half plane roots to $H_n(s)$ and the right-half plane roots to $H_n(-s)$.

3. Combine terms in the denominator of $H_n(s)$ to form first-and second-order factors.

$H_n(s)$ can be written in the following form:

$$|H_n(s)|^2 = \frac{1}{\prod_{\substack{LHP \\ poles}} (s - s_k)} = \frac{1}{B_n(s)} \tag{3.2}$$



For n odd For n even

Table (3.1): Butterworth Polynomials in Standard and Factored Forms

Standard form

$B_n(s) = a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + a_0$

a_8	a_7	a_6	a_5	a_4	a_3	a_2	a_1	a_0	n
							1	1	1
						1	$\sqrt{2}$	1	2
					1	2	2	1	3
				1	2.613	3.414	2.613	1	4
			1	3.236	5.236	5.236	3.236	1	5
		1	3.864	7.464	9.141	7.464	3.864	1	6
	1	4.494	10.103	14.606	14.606	10.103	4.494	1	7
1	5.126	13.138	21.848	25.691	21.848	13.138	5.126	1	8

Factored form

$B_n(s)$	n
$s + 1$	1
$s^2 + \sqrt{2}s + 1$	2
$(s^2 + s + 1)(s + 1)$	3
$(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)$	4
$(s + 1)(s^2 + 0.6180s + 1)(s^2 + 1.6180s + 1)$	5
$(s^2 + 0.5176s + 1)(s^2 + \sqrt{2}s + 1)(s^2 + 1.9318s + 1)$	6
$(s + 1)(s^2 + 0.4450s + 1)(s^2 + 1.2456s + 1)(s^2 + 1.8022s + 1)$	7
$(s^2 + 0.3986s + 1)(s^2 + 1.1110s + 1)(s^2 + 1.6630s + 1)(s^2 + 1.9622s + 1)$	8

Butterworth filter

$$H_n(s) = \frac{1}{a_n s^n + a_{n-1}s^{n-1} + \cdots + a_1 s + 1} = \frac{1}{B_n(s)}$$

Example 3.1: Find the transfer function $H_n(s)$ for the normalized Butterworth filter of order 2.

Solution: since $n = 2$ we have the poles of $H_2(s)\, H_2(-s)$ given by

$$s_k = 1\underline{/\pi/4 + k\pi/2}, \qquad k = 0, 1, 2, 3$$

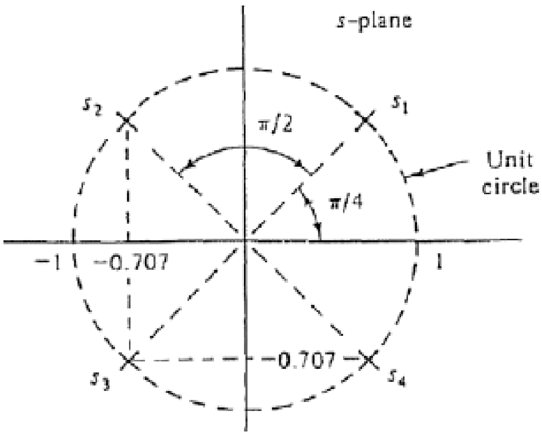
Therefore, the four roots are

$$s_1 = 1\angle 45^\circ = 1/\sqrt{2} + j/\sqrt{2},$$

$$s_2 = 1\angle 135^\circ = -1/\sqrt{2} + j/\sqrt{2},$$

$$s_3 = 1\angle 225^\circ = -1/\sqrt{2} + -j/\sqrt{2},$$

$$s_4 = 1\angle 315^\circ = 1/\sqrt{2} + -j/\sqrt{2}.$$



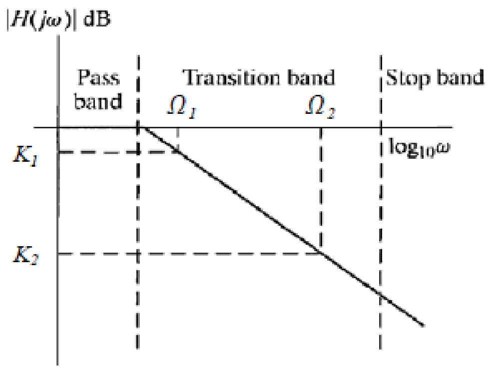
Using the left-half plane poles we can express the transfer function as follows

$$H_2(s) = \frac{1}{(s - s_2)(s - s_3)}$$

$$= \frac{1}{[s - (-0.707 - 0.707j)][s - (-0.707 + 0.707j)]}$$

$$= \frac{1}{s^2 + \sqrt{2}s + 1}$$

3.2 The Order of a Butterworth Filter



In the design of a low-pass filter, the filtering specifications are usually given in terms of the abruptness of the transition region, as shown in Figure beside. Once K_1 , Ω_1 , K_2 and Ω_2 are specified, the order of the Butterworth filter,

$$K_1 = 20\, log_{10} \frac{1}{\sqrt{1 + \left(\frac{\Omega_1}{\Omega_c}\right)^{2n}}}$$

$$K_1 = -10\, log_{10} \left(1 + \left(\frac{\Omega_1}{\Omega_c}\right)^{2n}\right) \qquad (3.3)$$

Solution:

$$K_2 = 20\,log_{10}\frac{1}{\sqrt{1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n}}}$$

$$K_2 = -10\,log_{10}\left(1+\left(\frac{\Omega_2}{\Omega_c}\right)^{2n}\right) \qquad (3.4)$$

If we wish to satisfy our requirement of Ω_c at Ω_1 exactly and do better than our requirement at Ω_2 we use

$$\left(\frac{\Omega_1}{\Omega_c}\right)^{2n} = 10^{-0.1K_1} - 1 \qquad (3.5)$$

While if we wish to satisfy our requirement at Ω_2 and exceed our requirement at Ω_1 we use

$$\left(\frac{\Omega_2}{\Omega_c}\right)^{2n} = 10^{-0.1K_2} - 1 \qquad (3.6)$$

Dividing Eqn.(3.6) by (3.5) to cancel Ω_c we have

$$\left(\frac{\Omega_2}{\Omega_1}\right)^{2n} = \frac{10^{-0.1K_2} - 1}{10^{-0.1K_1} - 1} \qquad (3.7)$$

A simple closed form answer for is easily obtained from this expression and is given by

$$n = \left\lceil \frac{log_{10}[(10^{-0.1K_1} - 1)/(10^{-0.1K_2} - 1)}{2log_{10}\frac{\Omega_1}{\Omega_2}} \right\rceil \qquad (3.8)$$

Where is the next larger integer.

Example 3.2:

a) Determine the order of a Butterworth filter that has a cutoff frequency of 1000 Hz and a gain of no more than -50 dB at 6000 Hz.
b) What is the actual gain in dB at 6000 Hz?

Solution:

a) The critical requirements are

$$\Omega_1 = \Omega_c = 2\pi(1000) rad/s \qquad K_1 = 20log_{10}\left(\frac{1}{\sqrt{2}}\right) = -3dB$$

$$\Omega_2 = 2\pi(6000) rad/s \qquad K_2 \leq -50dB$$

Substituting these requirements into Eqn.(3.8) gives

$$n = \left\lceil \frac{log_{10}[(10^{-0.1K_1} - 1)/(10^{-0.1K_2} - 1)}{2log_{10}\frac{\Omega_1}{\Omega_2}} \right\rceil$$

$$n = \left\lceil \frac{log_{10}\left[(10^{-0.1(-3)} - 1\right)/(10^{-0.1(-50)} - 1)}{2log_{10}\left(\frac{2\pi(1000)}{2\pi(6000)}\right)} \right\rceil n = \lceil 3.21 \rceil = 4$$

Therefore, we need a 4^{th} order Butterworth filter.

b) We can use Eq.(3.4) to calculate the actual gain at 6000 Hz. The gain in decibels will be

$$K_{2(actual)} = 20\ log_{10} \frac{1}{\sqrt{1 + \left(\frac{2\pi(6000)}{2\pi(1000)}\right)^{2(4)}}} = -62.25\text{ dB}$$

Example 3.3:

a) Determine the order of a Butterworth filter whose magnitude is 10 dB or better less than the passband magnitude at 500 Hz and at least 60 dB less than the passband magnitude at 5000 Hz.
b) Determine the cutoff frequency of the filter (in hertz).
c) What is the actual gain of the filter (in decibels) at 5000 Hz?

Solution:

a) The critical requirements are

$$\Omega_1 = 2\pi(500) rad/s \qquad K_1 = -10dB$$

$$\Omega_2 = 2\pi(5000) rad/s \qquad\qquad K_2 \le -60dB$$

$$n = \left\lceil \frac{log_{10}[(10^{-0.1K_1} - 1)/(10^{-0.1K_2} - 1)}{2log_{10}\frac{\Omega_1}{\Omega_2}} \right\rceil$$

$$n = \left\lceil \frac{log_{10}\left[(10^{-0.1(-10)} - 1\right)/(10^{-0.1(-60)} - 1)}{2log_{10}\left(\frac{500}{5000}\right)} \right\rceil = \lceil 2.52 \rceil = 3$$

Therefore we need a 3^{rd} order Butterworth filter to meet the specifications.

b) To do better at 500 Hz, we have to use Eq. 3.5, to determine the cutoff frequency.

$$\left(\frac{2\pi(500)}{\Omega_c}\right)^{2(3)} = 10^{-0.1(10)} - 1$$

Then, $\Omega_c = 2178.26 \frac{rad}{s} (f_c = 346.68\ Hz)$

c) The actual gain of the filter at 5000 Hz is

$$K_{2(actual)} = 20\ log_{10} \frac{1}{\sqrt{1 + \left(\frac{5000}{346.68}\right)^{2(3)}}} = -69.54 \text{ dB}$$

<u>3.3 Analog-to-Analog Transformations</u>

If we replace s of $H(s)$, the system function for a normalized low-pass filter, by s/Ω_u, we get a new transfer function $H'(s)$,given by

$$H'(s) = H(s)|_{s \to s/\Omega_u} = H(s/\Omega_u)$$

If we evaluate the magnitude of the transfer function at to get the frequency response we have

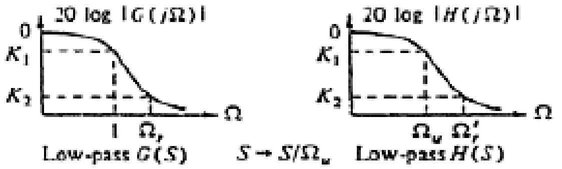
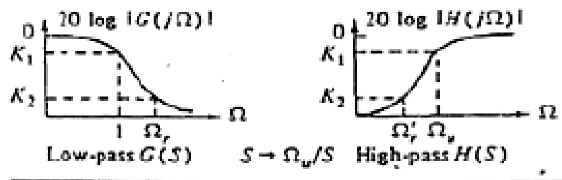
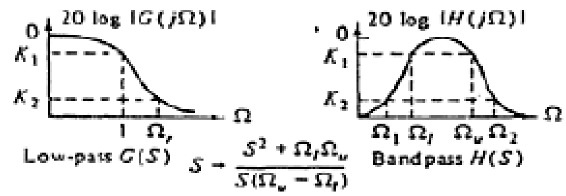
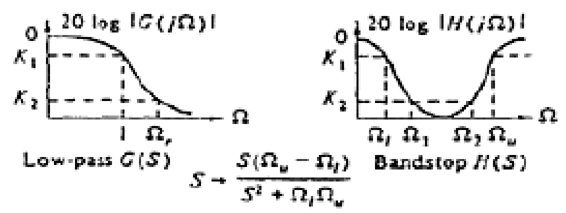
$$|H'(j\Omega)| = |H(j\Omega/\Omega_u)|$$

At the value of $\Omega = \Omega_u$ we have

$$|H'((j\Omega_u)| = |H(j\Omega_u/\Omega_u)| = |H(j1)|$$

That is, the frequency response for the new transfer function evaluated $\Omega = \Omega_u$ at is equal to the value of the normalized transfer function at $\Omega = 1$. In a sense we have moved the cut off frequency from $1\,rad/s\ to\ \Omega_u$ and thus have a scaling of the frequency axis. Similar transformations can be defined for taking low-pass transfer functions to high-pass, bandpass and bandstop transfer functions. Table (3.2) gives these transformations.

TABLE 3.2 ANALOG-TO-ANALOG TRANSFORMATION

Prototype response / Transformed filter response	Design equations
	Forward: $\Omega'_r = \Omega_r\Omega_u$ Backward: $\Omega_r = \Omega'_r/\Omega_u$
	Forward: $\Omega'_r = \Omega_u/\Omega_r$ Backward: $\Omega_r = \Omega_u/\Omega'_r$
	Forward: $\Omega_{xv} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = (\Omega_r^2\Omega_{xv}^2 + \Omega_l\Omega_u)^{1/2} - \Omega_{xv}\Omega_r$ $\Omega_2 = (\Omega_r^2\Omega_{xv}^2 + \Omega_l\Omega_u)^{1/2} + \Omega_{xv}\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = (-\Omega_1^2 + \Omega_l\Omega_u)/[\Omega_1(\Omega_u - \Omega_l)]$ $B = (+\Omega_2^2 - \Omega_l\Omega_u)/[\Omega_2(\Omega_u - \Omega_l)]$
	Forward: $\Omega_{xv} = (\Omega_u - \Omega_l)/2$ $\Omega_1 = [(\Omega_{xv}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} - \Omega_{xv}/\Omega_r$ $\Omega_2 = [(\Omega_{xv}/\Omega_r)^2 + \Omega_l\Omega_u]^{1/2} + \Omega_{xv}/\Omega_r$ Backward: $\Omega_r = \min\{\lvert A\rvert, \lvert B\rvert\}$ $A = \Omega_1(\Omega_u - \Omega_l)/[-\Omega_1^2 + \Omega_l\Omega_u]$ $B = \Omega_2(\Omega_u - \Omega_l)/[-\Omega_2^2 + \Omega_l\Omega_u]$

<u>Example 3.4:</u> Design an analog Butterworth filter that has a -2 dB or better cutoff frequency of 20 rad/sec and at least 10 dB of attenuation at 30 rad/sec.

<u>Solution:</u> the critical requirements are

$$\Omega_1 = 20\ ,\quad K_1 = -2\quad ,\quad \Omega_2 = 30\quad ,\quad K_2 = -10$$

$$n = \left\lceil \frac{log_{10}[(10^{-0.1K_1} - 1)/(10^{-0.1K_2} - 1)}{2log_{10}\frac{\Omega_1}{\Omega_2}} \right\rceil$$

$$n = \left\lceil \frac{log_{10}\big[(10^{-0.1(-2)} - 1\big)/(10^{-0.1(-10)} - 1)}{2log_{10}\left(\frac{20}{30}\right)} \right\rceil = \lceil 3.3709 \rceil = 4$$

Using this value of n to exactly satisfy the -2 dB requirement gives

$$\Omega_c = 20/(10^{-0.1(-2)} - 1)^{1/8} = 21.3868$$

The normalized low-pass Butterworth filter for $n = 4$, can be found from Table (3.1) as

$$H_4(s) = \frac{1}{(s^2 + 0.76536s + 1)(s^2 + 1.84776s + 1)}$$

Applying a low-pass to low-pass transformation, $s \to s/\Omega_c$, with $\Omega_c = 21.3868$ gives the desired transfer function as follows:

$$H(s) = H_4(s)\big|_{s \to \frac{s}{21.3868}}$$

$$= \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 0.76536\left(\frac{s}{21.3868}\right) + 1\right]}$$

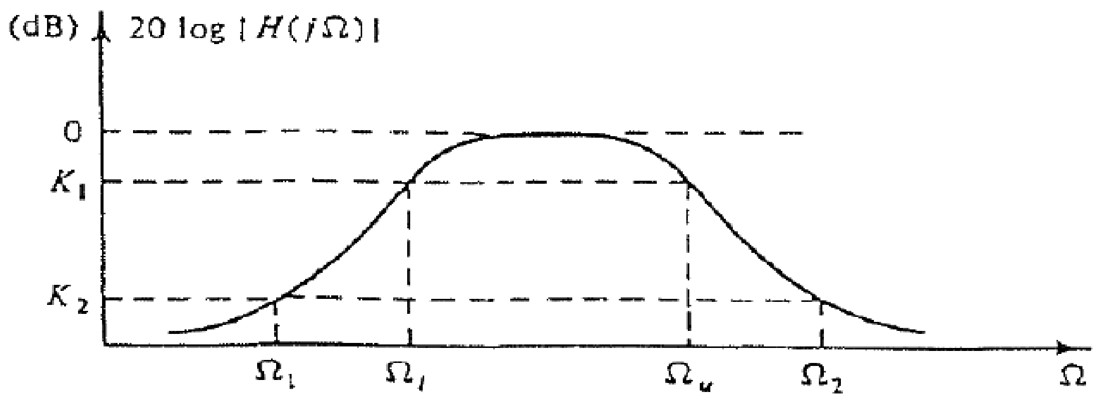
$$\times \frac{1}{\left[\left(\frac{s}{21.3868}\right)^2 + 1.84776\left(\frac{s}{21.3868}\right) + 1\right]}$$

$$= \frac{2.09210 \times 10^5}{(s^2 + 16.3686s + 457.394)(s^2 + 39.5176s + 457.394)}$$

3.4 Design of Bandpass Butterworth filter

The procedures for the design of a bandpass filter $H_{BP}(s)$, to satisfy the given set of specifications is composed of two steps.

1. Design a low-pass filter $H_{LP}(s)$ with Ω_r,
2. Apply the low-pass to bandpass transformation using the desired Ω_u and Ω_l.



Example 3.5: Design an analog bandpass filter with the following characteristics:

(a) -3.0103 dB upper and lower cutoff frequency of 20 kHz and 50 Hz respectively

(b) A stopband attenuation of at least 20 dB at 20 Hz and 45 kHz.

Solution: From the specifications above we can identify the following critical frequencies:

$$\Omega_1 = 2\pi(20) = 125.663\, rad/sec$$

$$\Omega_2 = 2\pi(45) = 2.82743 \times\ 10^5 rad/sec$$

$$\Omega_u = 2\pi(20) = 1.25663 \times\ 10^5\ rad/sec$$

$$\Omega_l = 2\pi(50) = 314.159\, rad/sec$$

Also the low-pass prototype must satisfy

$$0 \geq 20log|H_{LP}(j1)| \geq -3.0103\ dB$$

$$20log|H_{LP}(j\Omega_r)| \leq -20\ dB$$

From Table (3.2)

$$A = 2.5053$$

$$B = 2.2545$$

Since,

$$\Omega_\mathrm{r} = \min\{|\mathrm{A}|, |\mathrm{B}|\}$$

$$\Omega_\mathrm{r} = 2.2545$$

The low-pass Butterworth filter of order n is

$$n = \left|\left[\log\left(\frac{10^{0.30102}-1}{10^2-1}\right)\right] \Big/ \left[2\log\left(\frac{1}{2.2545}\right)\right]\right| = \lceil 2.829 \rceil = 3$$

From the Butterworth Table (3.1) and n we have the low-pass prototype as

$$H_{LP} = \frac{1}{s^3 2 + 2s^2 + 2s + 1}$$

The required analog-to-analog transformation is determined from Ω_u and Ω_l as

$$s \to \frac{s^2 + \Omega_l\Omega_u}{s(\Omega_u - \Omega_l)} = \frac{s^2 + 3.94784 \times\ 10^7}{s(1.25349 \times\ 10^5)}$$

$H_{BP}(s)$ then is finally seen to be

$$H_{BP}(s) = \frac{1}{\left\{\begin{array}{c}\left[\frac{s^2+3.94784\times 10^7}{s(1.25349\times 10^5)}\right]^3 + 2\left[\frac{s^2+3.94784\times 10^7}{s(1.25349\times 10^5)}\right]^2 \\ +2\frac{s^2+3.94784\times 10^7}{s(1.25349\times 10^5)} + 1\end{array}\right\}}$$

$$H_{BP}(s) = \frac{1.969530\times 10^{15}s^3}{\left\{\begin{array}{c}S^6 + 2.5069909\times 10^5 s^5 + 3.15434\times 10^{10}s^4 \\ +1.9893\times 10^{15}s^3 + 1.245285\times 10^{18}s^2 \\ +3.9072593\times 10^{20}s + 6.15289108\times 10^{22}\end{array}\right\}}$$

3.5 Chebyshev Filters

Chebyshev filters are defined in terms of the Chebyshev polynomials:

$$T_n(x) = \begin{cases}\cos(n\cos^{-1}x) & |x| \le 1 \\ \cosh(n\cosh^{-1}x) & |x| > 1\end{cases} \qquad (3.9a)$$

These polynomials may be generated recursively as follows,

$$T_n(x) = 2xT_{n-1}(x) - T_{n-2}(x) \qquad n \ge 2 \qquad (3.9b)$$

With $T_0(x) = 1$ and $T_1(x) = x$. a list of the first seventh Chebyshev polynomials is given in Table (3.3) for reference.

Table (3.3) the first seventh Chebyshev polynomials

n	$T_n(x)$
0	1
1	x
2	$2x^2-1$
3	$4x^3-3x$
4	$8x^4-8x^2+1$
5	$16x^5-20x^3+5x$
6	$32x^6-48x^4+18x^2-1$
7	$64x^7-112x^5+56x^3-7x$

The following properties of the Chebyshev polynomials follow from Eqn.(3.9).

1. For $|x| \le 1$ the polynomials are bounded by 1 in magnitude, $|T_n(x)| \le 1$, and oscillate between ± 1. For $|x| > 1$, the polynomials increase monotonically with x.
2. $T_n(1) = 1$ for all n .

3. $T_n(0) = \pm 1$ for n even, and $T_n(0) = 0$ for n odd.

4. All of the roots of $T_n(x)$ are in the interval $-1 \le x \le 1$.

The magnitude square of the frequency response for type I Chebyshev filter is

$$|H_n(j\Omega)|^2 = \frac{1}{1 + \epsilon^2 T_n^2(\Omega)}$$

Where n is the order of the filter, and ϵ is parameter that controls the passband ripple amplitude. Because $T_n^2(\Omega)$varies between 0 and 1 for $|\Omega| < 1$, $|H_n(j\Omega)|^2$ oscillate between 1 and $1/(1+\epsilon^2)$. As the order of the filter increases, the number of oscillations (ripples) in the passband increases, and the transition width between the passband and stopband becomes narrower. Example are given in Fig.(3.3) for $n = 5$ and 6.

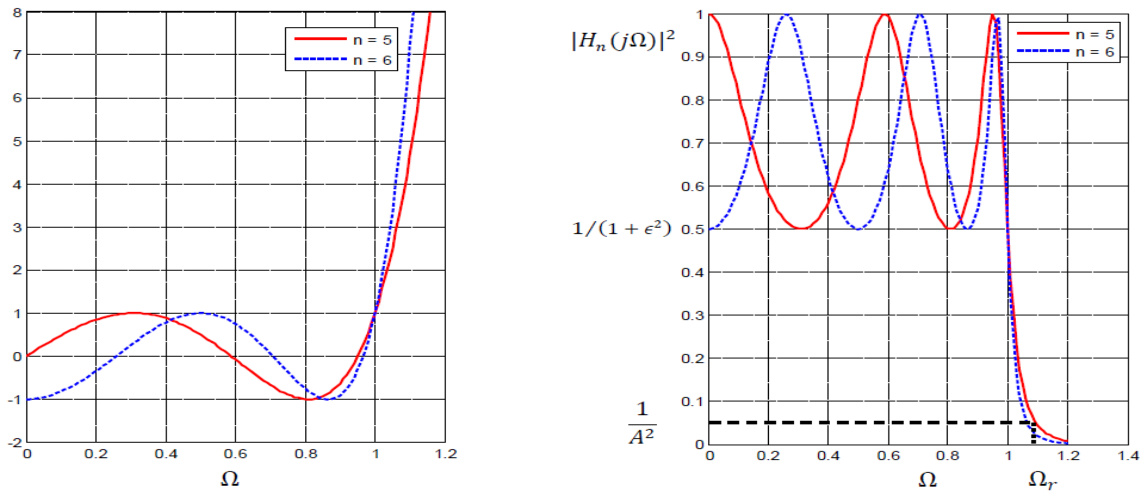


Fig.(3.3)

Form the rational function

$$|H_n(s)|^2 = H_n(s)H_n(-s) = \frac{1}{1 + \epsilon^2 T_n^2(s/j)}$$

Construct the system function $H_n(s)$ by taking the n poles that lie in the left-half s-plane.

$$H_n(s) = \frac{K}{\prod_{\substack{LHP \\ poles}} (s - s_k)} = \frac{K}{V_n(s)}$$

Where K is a normalizing factor whose value makes $H_n(0)$ equal 1 for n odd and $1/\sqrt{1+\epsilon^2}$for n even.

$$K = V_n(0) = b_o \qquad\qquad n \quad odd$$

$$K = \frac{V_n(0)}{\sqrt{(1+\epsilon^2)}} \qquad n \quad even$$

$$V_n(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_n s + b_0$$

The order n that satisfied ripple characterized by ϵ and a stopband gain $1/A$ at a particular Ω_r is given by

$$n = \left\lceil \frac{log_{10}\left[g + (g^2-1)^{1/2}\right]}{log_{10}\left[\Omega_r + \left(\Omega_r{}^2 - 1\right)^{1/2}\right]} \right\rceil$$

Where

$$A = 1/|H_n(j\Omega_r)|$$

$$g = [(A^2-1)/\epsilon]^{1/2}$$

Table(3.4)gives the $V_n(s)$ in polynomial form $n = 1$ for to 10 and corresponding to 0.5, 1, 2 and 3dB ripples.

Table(3.4)

POLYNOMIALS $V_n(s)$ USED IN CHEBYSHEV FILTER DESIGN FOR $\frac{1}{2}$, 1-, 2-, and 3-dB RIPPLES.

$$\text{Chebyshev filter } H_n(s) = \frac{K_n}{V_n(s)} \qquad K_n = \begin{cases} b_0/(1+\epsilon^2)^{1/2} & \text{for } n \text{ even} \\ b_0 & \text{for } n \text{ odd} \end{cases}$$

$$V_n(s) = s^n + b_{n-1}s^{n-1} + \cdots + b_1 s + b_0$$

n	b_0	b_1	b_2	b_3	b_4	b_5	b_6	b_7	b_8	b_9
				a. $\frac{1}{2}$ dB Ripple($\epsilon = 0.3493114$, $\epsilon^2 = 0.1220184$)						
1	2.8627752									
2	1.5162026	1.4256245								
3	0.7156938	1.5348954	1.2529130							
4	0.3790506	1.0254553	1.7168662	1.1973856						
5	0.1789234	0.7525181	1.3095747	1.9373675	1.1724909					
6	0.0947626	0.4323669	1.1718613	1.5897635	2.1718446	1.1591761				
7	0.0447309	0.2820722	0.7556511	1.6479029	1.8694079	2.4126510	1.1512176			
8	0.0236907	0.1525444	0.5735604	1.1485894	2.1840154	2.1492173	2.6567498	1.1460801		
9	0.0111827	0.0941198	0.3408193	0.9836199	1.6113880	2.7814990	2.4293297	2.9027337	1.1425705	
10	0.0059227	0.0492855	0.2372688	0.6269689	1.5274307	2.1442372	3.4409268	2.7097415	3.1498757	1.1400664
				b. 1-dB Ripple ($\epsilon = 0.5088471$, $\epsilon^2 = 0.2589254$)						
1	1.9652267									
2	1.1025103	1.0977343								
3	0.4913067	1.2384092	0.9883412							
4	0.2756276	0.7426194	1.4539248	0.9528114						
5	0.1228267	0.5805342	0.9743961	1.6888160	0.9368201					
6	0.0689069	0.3070808	0.9393461	1.2021409	1.9308256	0.9282510				
7	0.0307066	0.2136712	0.5486192	1.3575440	1.4287930	2.1760778	0.9231228			
8	0.0172267	0.1073447	0.4478257	0.8468243	1.8369024	1.6551557	2.4230264	0.9198113		
9	0.0076767	0.0706048	0.2441864	0.7863109	1.2016071	2.3781188	1.8814798	2.6709468	0.9175476	
10	0.0043067	0.0344971	0.1824512	0.4553892	1.2444914	1.6129856	2.9815094	2.1078524	2.9194657	0.9159320
				c. 2-dB Ripple ($\epsilon = 0.7647831$ $\epsilon^2 = 0.5848932$)						
1	1.3075603									
2	0.6367681	0.8038164								
3	0.3268901	1.0221903	0.7378216							
4	0.2057651	0.5167981	1.2564819	0.7162150						
5	0.0817225	0.4593491	0.6934770	1.4995433	0.7064606					
6	0.0514413	0.2102706	0.7714618	0.8670149	1.7458587	0.7012257				
7	0.0204228	0.1660920	0.3825056	1.1444390	1.0392203	1.9935272	0.6978929			
8	0.0128603	0.0729373	0.3587043	0.5982214	1.5795807	1.2117121	2.2422529	0.6960646		
9	0.0051076	0.0543756	0.1684473	0.6444677	0.8568648	2.0767479	1.3837464	2.4912897	0.6946793	
10	0.0032151	0.0233347	0.1440057	0.3177560	1.0389104	1.1585287	2.6362507	1.5557424	2.7406032	0.6936904
				d. 3-dB Ripple ($\epsilon = 0.9976283$ $\epsilon^2 = 0.9952623$)						
1	1.0023773									
2	0.7079478	0.6448996								
3	0.2505943	0.9283480	0.5972404							
4	0.1769869	0.4047679	1.1691176	0.5815799						
5	0.0626391	0.4079421	0.5488626	1.4149847	0.5744296					
6	0.0442467	0.1634299	0.6990977	0.6906098	1.6628481	0.5706979				
7	0.0156621	0.1461530	0.3000167	1.0518448	0.8314411	1.9115507	0.5684201			
8	0.0110617	0.0564813	0.3207646	0.4718990	1.4666990	0.9719473	2.1607148	0.5669476		
9	0.0039154	0.0475900	0.1313851	0.5834984	0.6789075	1.9438443	1.1122863	2.4101346	0.5659234	
10	0.0027654	0.0180313	0.1277560	0.2492043	0.9499208	0.9210659	2.4834205	1.2526467	2.6597378	0.5652718

n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n = 10
a. 1/2-dB Ripple (ϵ = 0.3493114, ϵ^2 = 0.1220184)									
−2.8627752	−0.7128122 ±j1.0040425	−0.6264565	−0.1753531 ±j1.0162529	−0.3623196	−0.0776501 ±j1.0084608	−0.2561700	−0.0436201 ±j1.0050021	−0.1984053	−0.0278994 ±j1.0032732
		−0.3132282 ±j1.0219275	−0.4233398 ±j0.4209457	−0.1119629 ±j1.0115574	−0.2121440 ±j0.7382446	−0.0570032 ±j1.0064085	−0.1242195 ±j0.8519996	−0.0344527 ±j1.0040040	−0.0809672 ±j0.9050658
				−0.2931227 ±j0.6251768	−0.2897940 ±j0.2702162	−0.1597194 ±j0.8070770	−0.1859076 ±j0.5692879	−0.0992026 ±j0.8829063	−0.1261094 ±j0.7182643
						−0.2308012 ±j0.4478939	−0.2192929 ±j0.1999073	−0.1519373 ±j0.6553170	−0.1589072 ±j0.4611541
								−0.1864400 ±j0.3486869	−0.1761499 ±j0.1589029
b. 1-dB Ripple (ϵ = 0.5088471, ϵ^2 = 0.2589254)									
−1.9652267	−0.5488672 ±j0.8951286	−0.4941706	−0.1395360 ±j0.98333792	−0.2894933	−0.0621810 ±j0.9934115	−0.2054141	−0.0350082 ±j0.9964513	−0.1593305	−0.0224144 ±j0.9977755
		−0.2470853 ±j0.9659987	−0.3368697 ±j0.4073290	−0.0894584 ±j0.9901071	−0.1698817 ±j0.7272275	−0.0457089 ±j0.9952839	−0.0996950 ±j0.8447506	−0.0276674 ±j0.9972297	−0.1013166 ±j0.7143234
				−0.2342050 ±j0.6119198	−0.2320627 ±j0.2661837	−0.1280736 ±j0.7981557	−0.1492041 ±j0.5644443	−0.0796652 ±j0.8769490	−0.0650493 ±j0.9001063
						−0.1850717 ±j0.4429430	−0.1759983 ±j0.1982065	−0.1220542 ±j0.6508954	−0.1276664 ±j0.4586271
								−0.1497217 ±j0.3463342	−0.1415193 ±j0.1580321

n = 1	n = 2	n = 3	n = 4	n = 5	n = 6	n = 7	n = 8	n = 9	n = 10
c. 2-dB Ripple (ϵ = 0.7647831, ϵ^2 = 0.5848932)									
−1.3075603	−0.4019082 ±0.6893750	−0.3689108-	−0.1048872 ±j0.9579530	−0.2183083	−0.0469732 ±j0.9817052	−0.1552958	−0.0264924 ±j0.9897870	−0.1206293	−0.0169758 ±j0.9934868
		−0.1844554 ±j0.9230771	−0.2532202 ±j0.3967971	−0.0674610 ±j0.9734557	−0.1283332 ±j0.7186581	−0.0345566 ±j0.9866139	−0.0754439 ±j0.8391009	−0.0209471 ±j0.9919471	−0.0767332 ±j0.7112580
				−0.1766151 ±j0.6016287	−0.1753064 ±j0.2630471	−0.0968253 ±j0.7912029	−0.1129098 ±j0.5606693	−0.0603149 ±j0.8723036	−0.0492657 ±j0.8962374
						−0.1399167 ±j0.4390845	−0.1331862 ±j0.1968809	−0.0924078 ±j0.6474475	−0.0966894 ±j0.4566553
								−0.1133549 ±j0.3444996	−0.1071810 ±j0.1573528
d. 3-dB Ripple (ϵ = 0.9976283, ϵ^2 = 0.9952623)									
1.0023773	−0.3224498 ±j0.7771576	−0.2986202	−0.0851704 ±j0.9464844	−0.1775085	−0.0382295 ±j0.9764060	−0.1264854	−0.0215782 ±j0.9867664	−0.0982716	−0.0138320 ±j0.9915418
		−0.1493101 ±j0.9038144	−0.2056195 ±j0.3920467	−0.0548531 ±j0.9659238	−0.1044450 ±j0.7147788	−0.0281456 ±j0.9826957	−0.0614494 ±j0.8365401	−0.0170647 ±j0.9895516	−0.0401419 ±j0.8944827
				−0.1436074 ±j0.5969738	−0.1426745 ±j0.2616272	−0.0788623 ±j0.7880608	−0.0919655 ±j0.5589582	−0.0491358 ±j0.8701971	−0.0625225 ±j0.7098655
						−0.1139594 ±j0.4373407	−0.1084807 ±j0.1962800	−0.0752804 ±j0.6458339	−0.0787829 ±j0.4557617
								−0.0923451 ±j0.3436677	−0.0873316 ±j0.1570443

Example 3.6: Design a Chebyshev lowpass filter to satisfy the following specifications:

(a) Acceptable passband ripple of 2 dB.

(b) Cutoff radian frequency of 40 rad/sec.

(c) Stopband attenuation of 20 dB or more at 52 rad/sec.

***Solution*:** the general approach is

(a) Change the requirements to those of a lowpass unit bandwidth prototype;

$$\Omega_r = \frac{\Omega_r{}'}{\Omega_u} = \frac{52}{40} = 1.3$$

(b) Design a normalized Chebyshev lowpass filter with cutoff at 1rad/sec and $\Omega_r = 1.3$

$$20log|H_n(j1)| = 20log\left[1/\sqrt{1+\epsilon^2}\right] = -2 \quad \rightarrow \epsilon = 0.76478$$

$$20log|H_n(j1.3)| = 20log[1/A] = -20 \quad \rightarrow A = 10$$

$$g = [(100-1)/(0.76478)^2]^{1/2} = 13.01$$

$$n = \left\lceil \frac{log_{10}\left[g + (g^2-1)^{1/2}\right]}{log_{10}\left[\Omega_r + \left(\Omega_r{}^2 - 1\right)^{1/2}\right]} \right\rceil$$

$$n = \left\lceil \frac{log_{10}\left[13.01 + ((13.01)^2-1)^{1/2}\right]}{log_{10}[1.3 + ((1.3)^2-1)^{1/2}]} \right\rceil = \lceil 4.3 \rceil = 5$$

Using the 2-dB ripple part of Table (3.4) for $n = 5, H(0) = 1$ $(n\ odd)$, the desired Chebyshev unit bandwidth lowpass filter is

$$H_5(s) = K/(s^5 + b_4 s^4 + b_3 s^3 + b_2 s^2 + b_1 s + b_0$$

$$H_5(s) = \frac{0.08172}{s^5 + 0.70646s^4 + 1.4995s^3 + 0.6934s^2 + 0.459349s + 0.08172}$$

(c) From Table (3.2), apply the transformation $s \to s/40$

$$H_d(s) = H_5(s)|_{s \to \frac{s}{40}}$$

$$= \frac{0.08172}{\left(\frac{s}{40}\right)^5 + 0.70646s^4 + 1.4995\left(\frac{s}{40}\right)^3 + 0.6934\left(\frac{s}{40}\right)^2 + 0.459349\left(\frac{s}{40}\right) + 0.08172}$$

$$= \frac{8.368128 \times 10^6}{s^5 + 28.2584s^4 + 2399.2s^3 + 44377.6s^2 + 1.17593344 \times 10^6 s + 8.368128 \times 10^6}$$

3.6 The Bilinear Transformation

The bilinear transformation is a mapping from the s-plane to the z-plane defined by

$$s = \frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}$$

Where $1/T$ is the sampling frequency which can be set to 1. Given an analog filter with a system function $H_d(s)$, the digital filter is designed as follows:

$$H(z) = H_d\left(\frac{2}{T}\frac{1 - z^{-1}}{1 + z^{-1}}\right)$$

The steps involved in the design of a digital low-pass filter with a passband cutoff frequency ω_1, stopband cutoff frequency ω_2, passband magnitude K_1 and stopband magnitude K_2 ,are as follows:

1. Prewar the passband and stopband cutoff frequencies of the digital filter, ω_1 and ω_2, using the following formula

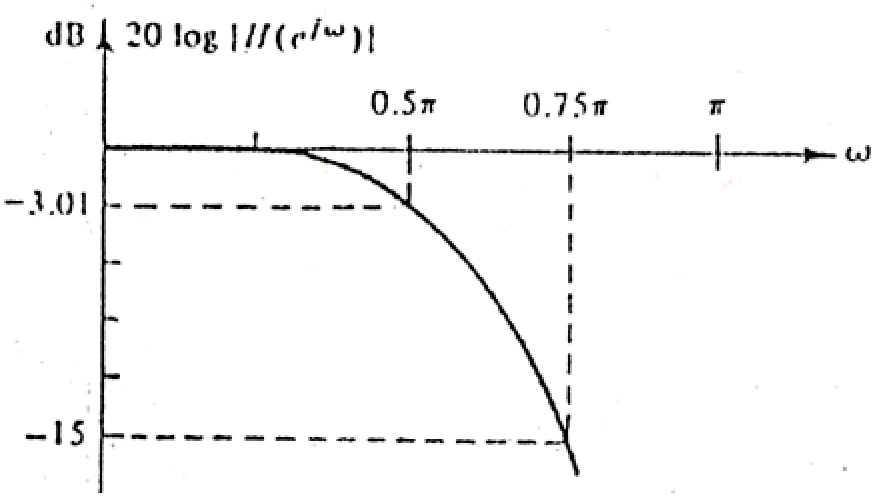
$$\Omega_i = \frac{2}{T}\tan\left(\frac{\omega_i}{2}\right) \qquad i = 1,2$$

   to determine the passband and cutoff frequencies of the analog low-pass filter.
2. Design an analog low-pass filter with the specifications Ω_1, Ω_2, K_1 and K_2.
3. Apply the bilinear transformation $T = 1$ to the filter designed in step2.

Example 3.7: Design and realize a digital low-pass filter using the bilinear transformation method to satisfy the following characteristics:

(a) -3.01 dB cutoff frequency of 0.5π rad.

(b) Magnitude down at least 15 dB at 0.75π rad. The required frequency response is shown below



Solution:

Step 1. Prewar critical frequencies using $T = 1$.

$$\Omega_i = \frac{2}{T}\tan\left(\frac{\omega_i}{2}\right) \qquad\qquad i = 1,2$$

$$\Omega_1 = 2\ \tan\left(\frac{0.5\pi}{2}\right) = 2.000$$

$$\Omega_2 = 2\ \tan\left(\frac{0.75\pi}{2}\right) = 4.8282$$

Step 2.Design an analog low-pass filter with the specifications Ω_1, Ω_2, K_1 and K_2.

$$n = \left\lceil \frac{log_{10}\left[\left(10^{3.01/10} - 1\right)/\left(10^{15/10} - 1\right)\right]}{2log_{10}\left(\frac{2}{4.8282}\right)} \right\rceil = \lceil 1.9412 \rceil = 2$$

Using this value of n to exactly satisfy the -3.01 dB requirement gives

$$\Omega_c = \frac{20}{(10^{3.01/10} - 1)^{\frac{1}{4}}} = 2$$

Therefore the required prewarped analog filter using the Butterworth Table (3.1) and the low-pass to low-pass transformation from Table (3.2) is

$$H_a(s) = \left.\frac{1}{s^2 + \sqrt{2}s + 1}\right|_{s \to s/2} = \frac{4}{s^2 + 2\sqrt{2}s + 4}$$

Step3 . Applying the bilinear transformation

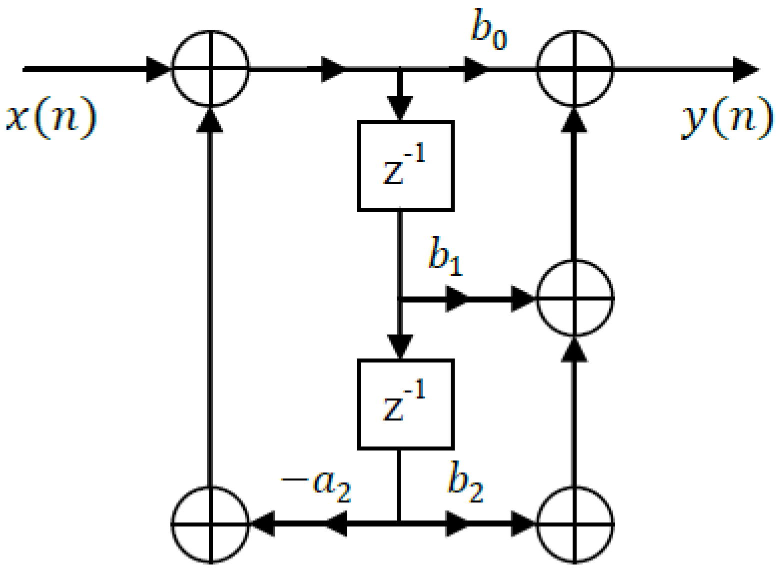
$$H(z) = H_a(s)\big|_{s=\frac{2(1-z^{-1})}{(1+z^{-1})}}$$

$$= \frac{4}{\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right]^2 + 2\sqrt{2}\left[\frac{2(1-z^{-1})}{(1+z^{-1})}\right] + 4}$$

$$= \frac{1 + 2z^{-1} + z^{-2}}{3.4142135 + 0.5857865z^{-2}}$$

$$= \frac{0.29289 + 0.58578z^{-1} + 0.29289z^{-2}}{1 + 0.17157z^{-2}}$$

Direct II realization of this filter is shown below, where $b_o = 0.29289, b_1 = 0.58578, b_2 = 0.29289, a_1 = 0, a_2 = 0.17157$



The plot of $20\ log\left|H(e^{jw})\right|$ versus ω is shown below